

# Hamiltonian and Brownian systems with long-range interactions:

## II. Kinetic equations and stability analysis

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### Abstract

We discuss the kinetic theory of systems with long-range interactions. We contrast the microcanonical description of an isolated Hamiltonian system described by the Liouville equation from the canonical description of a stochastically forced Brownian system described by the Fokker-Planck equation. We show that the mean-field approximation is exact in a proper thermodynamic limit. For  $N \rightarrow +\infty$ , a Hamiltonian system is described by the Vlasov equation. In this collisionless regime, coherent structures can emerge from a process of violent relaxation. These metaequilibrium states, or quasi-stationary states (QSS), are stable stationary solutions of the Vlasov equation. To order  $1/N$ , the collision term of a homogeneous system has the form of the Lenard-Balescu operator. It reduces to the Landau operator when collective effects are neglected. The statistical equilibrium state (Boltzmann) is obtained on a collisional timescale of order  $N$  or larger (when the Lenard-Balescu operator cancels out). We also consider the stochastic motion of a test particle in a bath of field particles and derive the general form of the Fokker-Planck equation describing the evolution of the velocity distribution of the test particle. The diffusion coefficient is anisotropic and depends on the velocity of the test particle. For Brownian systems, in the  $N \rightarrow +\infty$  limit, the kinetic equation is a non-local Kramers equation. In the strong friction limit  $\xi \rightarrow +\infty$ , or for large times  $t \gg \xi^{-1}$ , it reduces to a non-local Smoluchowski equation. We give explicit results for self-gravitating systems, two-dimensional vortices and for the HMF model. We also introduce a generalized class of stochastic processes and derive the corresponding generalized Fokker-Planck equations. We discuss how a notion of generalized thermodynamics can emerge in complex systems displaying anomalous diffusion.

Keywords: long-range interactions; mean-field theory; Hamiltonian systems; Brownian systems

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## I. INTRODUCTION

The statistical mechanics of systems with long-range interactions is currently a topic of active research [1]. In a previous paper [2] (paper I), we have considered general models of Hamiltonian and Brownian systems with long-range interactions, for an arbitrary binary potential of interaction in  $D$  dimensions, and we have studied their equilibrium properties. In the present paper, we discuss the kinetic equations describing the out-of-equilibrium evolution of these systems.

So far, most works have focused on the case of isolated Hamiltonian systems of particles in interaction such as the N-star problem in astrophysics [3, 4], the N-vortex problem in 2D hydrodynamics [4] and the HMF model [5, 6]. For these systems, the energy is conserved and the correct statistical ensemble is the *microcanonical* ensemble. In a proper thermodynamic limit  $N \rightarrow +\infty$ , the equilibrium one-body distribution function maximizes the Boltzmann entropy at fixed mass and energy. The “collisional” evolution of the distribution function is described by non-local kinetic equations such as the Landau-Poisson system (or the orbit averaged Fokker-Planck equation) in astrophysics [7] or the kinetic equations derived by Dubin & O’Neil [8] and Chavanis [9] for non-neutral plasmas and 2D point vortices. These equations increase the Boltzmann entropy  $S_B$  at fixed mass  $M$  (or circulation  $\Gamma$ ) and energy  $E$ . For stellar systems, the relaxation time (Chandrasekhar time) scales as  $t_{relax} \sim \frac{N}{\ln N} t_D$ , where  $t_D$  is the dynamical time [7]. For the point vortex gas, the evolution is due to a condition of resonance which can be satisfied only if the profile of angular velocity is non-monotonic [8, 9]. When the profile of angular velocity is monotonic, the distribution is stationary on a timescale of at least  $N t_D$  and it is not clear whether the system truly relaxes towards statistical equilibrium for longer times. The kinetic theory of the HMF model is also complicated [6, 10, 11] and seems to indicate a relaxation time scaling as  $t_{relax} \sim N^{1.7} t_D$  [12]. In the “collisionless” regime, valid for sufficiently “short” times  $t \ll t_{relax}$ , the above-mentioned kinetic equations reduce to the Vlasov-Poisson and 2D Euler-Poisson systems. Since the relaxation time  $t_{relax}$  increases rapidly (algebraically) with the number of particles, the Vlasov regime can be extremely long in practice (e.g., in astrophysics). The Vlasov-Poisson and 2D Euler-Poisson systems can undergo a phenomenon of collisionless relaxation (called violent relaxation in astrophysics) towards a metaequilibrium state on the coarse-grained scale [13, 14, 15, 16]. Since the relaxation time depends on  $N$ , the limits  $N \rightarrow +\infty$ ,  $t \rightarrow +\infty$  (metaequilibrium) and  $t \rightarrow +\infty$ ,  $N \rightarrow +\infty$  (statistical equilibrium) differ. The metaequilibrium state is described by non standard distribution functions which are nonlinearly dynamically stable stationary solutions of the Vlasov equation. In some cases, they maximize a H-function at fixed mass and energy [17, 18]. This maximization problem provides a refined criterion of nonlinear dynamical stability for the Vlasov-Poisson and 2D Euler-Poisson systems [6, 12, 19, 20, 21, 22, 23]. The statistical equilibrium state, reached for longer times, is described by the Boltzmann distribution.

Since statistical ensembles are not equivalent for systems with long-range interactions, it is of interest, at a conceptual level, to introduce a canonical model of particles with long-range interactions. In that respect, we can consider a system of Brownian particles described by  $N$ -coupled stochastic equations involving a friction and a random force in addition to long-range forces. These particles are in contact with a thermal bath that imposes the temperature  $T$ . For these systems, the correct statistical ensemble is the *canonical* ensemble. In a proper thermodynamic limit  $N \rightarrow +\infty$ , the equilibrium one-body distribution function minimizes the Boltzmann free energy  $F_B = E - T S_B$  at fixed mass  $M$  and temperature

*T.* The evolution of the distribution function of these Brownian particles is described by non-local Fokker-Planck equations which are the canonical counterpart of the kinetic equations governing the evolution of Hamiltonian systems. For example, a gas of self-gravitating Brownian particles [24] is governed by the Kramers-Poisson and Smoluchowski-Poisson systems. These equations decrease the Boltzmann free energy at fixed mass and temperature. Self-gravitating Brownian particles can experience an “isothermal collapse” [25], which is the canonical version of the “gravothermal catastrophe” [26] experienced by globular clusters. A Brownian model has also been introduced in the case of a cosinusoidal potential of interaction in  $d = 1$  [6]. This is the canonical counterpart of the microcanonical HMF model. It could be called the BMF (Brownian Mean Field) model.

In this paper, we compare these two descriptions (Hamiltonian and Brownian) and study their out-of-equilibrium properties. In Sec. II, we discuss the kinetic theory of Hamiltonian systems with long-range interactions by adapting the results of plasma physics to this more general context. In Sec. II A, starting from the Liouville equation, we consider a truncation of the BBGKY hierarchy in powers of the inverse particle number  $1/N$ , which plays the same role as the plasma parameter in plasma physics. For  $N \rightarrow +\infty$ , the kinetic equation is the Vlasov equation. For long-range interactions, this equation is non-local due to mean-field effects and exhibits a “violent relaxation” towards a metaequilibrium state on a few dynamical times  $t_D$  (Sec. II B). In Sec. II C, we study the linear dynamical stability of a spatially homogeneous solution of the Vlasov equation and derive a criterion of stability generalizing the Jeans criterion in astrophysics. We also determine analytical expressions for the growth rate and damping rates of the perturbation. In Sec. II E, we study the nonlinear dynamical stability of a stationary solution of the Vlasov equation and compare with the Euler equation (Sec. II D). We also show that, for a spatially homogeneous solution, the criterion of nonlinear stability coincides with the criterion of linear stability. In the sequel of the paper, we consider a spatially homogeneous system which is stable with respect to the Vlasov equation and we study the time evolution of its velocity distribution function due to finite  $N$  effects (collisions). To order  $1/N$ , the kinetic equation of a homogeneous system is the Landau equation (Sec. II F) when collective effects are ignored and the Lenard-Balescu equation (Sec. II G) when collective effects are properly accounted for. For 2D and 3D systems, these equations converge towards the statistical equilibrium state (Maxwellian distribution) on a relaxation time  $t_{relax} \sim Nt_D$  (for gravitational systems, the relaxation time is  $(N/\ln N)t_D$  due to logarithmic divergences). For one-dimensional systems, the Lenard-Balescu operator cancels out so that the relaxation, due to three-body (or higher) correlations, is longer than  $Nt_D$  (a similar result is obtained for the point vortex gas in two dimensions [8, 9]). In Sec. II H, we consider the relaxation of a test particle in a bath of field particles and derive the general form of the Fokker-Planck equation. The diffusion coefficient is anisotropic and depends on the velocity. This is responsible for anomalous diffusion and for a slow relaxation of the high velocity tail of the distribution [11, 27]. We provide various explicit expressions of the diffusion coefficient and friction force for a thermal bath with Maxwellian distribution function (subsections II H 1 and II H 2) and for one dimensional systems with an arbitrary distribution of the bath (subsection II H 3). In Sec. III, we study the temporal correlation function of the force and show that each mode decreases exponentially rapidly with a decay rate which coincides with the damping rate derived in the linear dynamical stability analysis of the Vlasov equation (Sec. II C). In Sec. II J, we consider the time evolution of the spatial correlation function of the particles in the linear regime and compare with the equilibrium results obtained in Paper I. In Sec. III, we develop the kinetic theory of Brownian systems

with long-range interactions. Starting from the  $N$ -body Fokker-Planck equation and using a mean-field approximation valid at the thermodynamic limit  $N \rightarrow +\infty$ , we derive a non-local Kramers equation. In the strong friction limit  $\xi \rightarrow +\infty$ , or for large times  $t \gg \xi^{-1}$ , it reduces to a non-local Smoluchowski equation. In Sec. III C, we study the evolution of the spatial correlation function for a Brownian system in the linear regime. Finally, in Sec. IV, we introduce a generalized class of stochastic processes and derive the corresponding generalized Fokker-Planck equations. We show that they display anomalous diffusion and that they are associated with a notion of generalized thermodynamics in  $\mu$ -space.

One interest of our general study is to present a unified description of systems with long-range interactions (Hamiltonian, Brownian, fluids,...) and to see how the results depend on the form of the potential of interaction and on the dimension of space  $d$ . Explicit results are given for gravitational systems, two-dimensional vortices and for the HMF model. Thus, our study shows the analogies and differences between these systems by placing them into a more general perspective.

## II. KINETIC THEORY OF HAMILTONIAN SYSTEMS

### A. The BBGKY hierarchy

We wish to develop a kinetic theory of Hamiltonian systems with long-range interactions described by the  $N$ -body equations (I-1) of paper I in order to obtain the evolution of the one-body distribution function  $f(\mathbf{r}, \mathbf{v}, t) = NmP_1(\mathbf{r}, \mathbf{v}, t)$ . We will see that many results of plasma physics developed for the Coulombian potential can be extended to a more general context. We shall discuss how these results depend on the dimension of space and on the form of the potential of interaction. We shall also discuss how the results are affected by the existence of a critical point in the case of *attractive* potentials (see Paper I).

Starting from the Liouville equation (I-2), it is simple to construct the BBGKY hierarchy of equations for the reduced distribution functions,

$$\frac{\partial P_j}{\partial t} + \sum_{i=1}^j \mathbf{v}_i \frac{\partial P_j}{\partial \mathbf{r}_i} + \sum_{i=1}^j \sum_{k=1, k \neq i}^j \mathbf{F}(k \rightarrow i) \frac{\partial P_j}{\partial \mathbf{v}_i} + (N-j) \sum_{i=1}^j \int d^D \mathbf{x}_{j+1} \mathbf{F}(j+1 \rightarrow i) \frac{\partial P_{j+1}}{\partial \mathbf{v}_i} = 0. \quad (1)$$

The first two equations of this hierarchy are

$$\frac{\partial P_1}{\partial t} + \mathbf{v}_1 \frac{\partial P_1}{\partial \mathbf{r}_1} + (N-1) \frac{\partial}{\partial \mathbf{v}_1} \int \mathbf{F}(2 \rightarrow 1) P_2(\mathbf{x}_1, \mathbf{x}_2, t) d^D \mathbf{x}_2 = 0, \quad (2)$$

$$\frac{\partial P_2}{\partial t} + \mathbf{v}_1 \frac{\partial P_2}{\partial \mathbf{r}_1} + \mathbf{F}(2 \rightarrow 1) \frac{\partial P_2}{\partial \mathbf{v}_1} + (N-2) \frac{\partial}{\partial \mathbf{v}_1} \int \mathbf{F}(3 \rightarrow 1) P_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) d^D \mathbf{x}_3 + (1 \leftrightarrow 2) = 0, \quad (3)$$

where  $\mathbf{x} = (\mathbf{r}, \mathbf{v})$ . Introducing the decomposition (I-14)-(I-15), corresponding to the first terms of the Mayer expansion in plasma physics, they can be rewritten

$$\begin{aligned} \frac{\partial P_1}{\partial t} + \mathbf{v}_1 \frac{\partial P_1}{\partial \mathbf{r}_1} + N \frac{\partial P_1}{\partial \mathbf{v}_1} \int \mathbf{F}(2 \rightarrow 1) P_1(\mathbf{x}_2, t) d^D \mathbf{x}_2 \\ + N \frac{\partial}{\partial \mathbf{v}_1} \int \mathbf{F}(2 \rightarrow 1) P'_2(\mathbf{x}_1, \mathbf{x}_2, t) d^D \mathbf{x}_2 = 0, \end{aligned} \quad (4)$$

$$\begin{aligned}
& \frac{\partial P'_2}{\partial t} + \mathbf{v}_1 \frac{\partial P'_2}{\partial \mathbf{r}_1} + \mathbf{F}(2 \rightarrow 1) \frac{\partial P'_2}{\partial \mathbf{v}_1} + \mathbf{F}(2 \rightarrow 1) P_1(\mathbf{x}_2, t) \frac{\partial P_1}{\partial \mathbf{v}_1}(\mathbf{x}_1, t) \\
& + N \frac{\partial}{\partial \mathbf{v}_1} \int \mathbf{F}(3 \rightarrow 1) P'_2(\mathbf{x}_1, \mathbf{x}_2, t) P_1(\mathbf{x}_3, t) d^D \mathbf{x}_3 \\
& + N \frac{\partial}{\partial \mathbf{v}_1} \int \mathbf{F}(3 \rightarrow 1) P'_2(\mathbf{x}_2, \mathbf{x}_3, t) P_1(\mathbf{x}_1, t) d^D \mathbf{x}_3 \\
& + N \frac{\partial}{\partial \mathbf{v}_1} \int \mathbf{F}(3 \rightarrow 1) P'_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) d^D \mathbf{x}_3 + (1 \leftrightarrow 2) = 0,
\end{aligned} \tag{5}$$

where we have taken  $N - 1 \simeq N$  and  $N - 2 \simeq N$  for  $N \gg 1$ . We shall now consider the thermodynamic limit defined in Sec. II.B of paper I. For example, we can consider  $N \rightarrow +\infty$  in such a way that the interaction potential (coupling constant) scales as  $u_* \sim 1/N$ , while  $\beta \sim 1$ ,  $E/N \sim 1$  and  $V \sim R^D \sim 1$ . In that limit, the cumulant distribution functions  $P'_j$  scale as  $1/N^{j-1}$ . We can therefore consider an expansion of the correlation functions in powers of the inverse particle number  $1/N$ . This small parameter is the counterpart of the “plasma parameter” in plasma physics. Therefore, the methods of plasma physics can be applied in the present context with a different perspective.

## B. Vlasov equation and violent relaxation

For  $N \rightarrow +\infty$ , we get

$$P_2(\mathbf{x}_1, \mathbf{x}_2, t) = P_1(\mathbf{x}_1, t) P_1(\mathbf{x}_2, t) + O(1/N) \tag{6}$$

so that the mean-field approximation is exact in a proper thermodynamic limit. In that case, the first equation of the BBGKY hierarchy reduces to

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \langle \mathbf{F} \rangle \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \tag{7}$$

where we have introduced the mean-field force produced by the particles,

$$\langle \mathbf{F} \rangle(\mathbf{r}, t) = \int \mathbf{F}(1 \rightarrow 0) n(\mathbf{r}_1, t) d^D \mathbf{r}_1. \tag{8}$$

The Vlasov equation (7) expresses the conservation of the distribution function in  $\mu$ -space in the absence of “collisions”. In the case of long-range interactions, the Vlasov equation is coupled to Eq. (8). This coupling creates a complicated mixing process in phase space leading to a *metaequilibrium* state on a very short timescale, of the order of the dynamical time  $t_D$ . This process is called violent relaxation in astrophysics [13] or chaotic mixing. It explains how a collisionless system can reach a quasi-equilibrium state (on a coarse-grained scale) as a result of phase mixing driven by long-range interactions. This metaequilibrium state, or quasi-stationary state (QSS), is a particular stationary solution of the Vlasov equation. Since it results from a turbulent mixing, it is particularly robust and has therefore nonlinear stability properties with respect to the collisionless dynamics. For a given initial condition, Lynden-Bell [13] has tried to predict the metaequilibrium state reached by the system by resorting to a new type of statistical mechanics taking into account the particularities of the collisionless dynamics (Casimirs) [28]. Unfortunately, his theory does not always give the

good result (i.e., the distribution that is actually reached by the system) because relaxation is incomplete in general so that the system can be trapped in a stationary solution of the Vlasov equation which is not the most mixed state. This concept of *incomplete relaxation* [13] explains why galaxies are more confined than predicted by Lynden-Bell's statistical mechanics. In some cases, the metaequilibrium state turns out to maximize a H-function [17]:

$$H[f] = - \int C(f) d^D \mathbf{r} d^D \mathbf{v}, \quad (9)$$

where  $C$  is convex, at fixed mass  $M = \int \rho d^D \mathbf{r}$  and energy  $E = \frac{1}{2} \int f v^2 d^D \mathbf{r} d^D \mathbf{v} + \frac{1}{2} \int \rho \Phi d^D \mathbf{r}$ . In the context of violent relaxation, Tsallis functional  $H_q = -\frac{1}{q-1} \int (f^q - f) d^D \mathbf{r} d^D \mathbf{v}$  [29] is a particular H-function [18, 22, 23, 28], not an entropy, which occasionally, but not systematically, gives a good fit of the metaequilibrium state. Its maximization at fixed mass and energy leads to a particular class of nonlinearly dynamically stable stationary solutions of the Vlasov equation called stellar polytropes in astrophysics [23]. The same interpretation holds in 2D hydrodynamics [21, 30] and for the HMF model [6, 12]. We refer to [23, 28] for a more thorough discussion of these concepts.

### C. Linear dynamical stability

We consider here the linear dynamical stability of a spatially homogeneous system with respect to the Vlasov equation (7)-(8). Following the standard derivation of the stability criterion and writing  $\delta f \sim \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$  with  $\omega = \omega_r + i\lambda$ , the dispersion relation reads [6]:

$$\epsilon(\mathbf{k}, \omega) \equiv 1 - (2\pi)^D \hat{u}(\mathbf{k}) \int \frac{\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}}}{\mathbf{k} \cdot \mathbf{v} - \omega} d^D \mathbf{v} = 0 \quad (10)$$

where  $f(\mathbf{v})$  is the unperturbed distribution function and  $\epsilon(\mathbf{k}, \omega)$  is the dielectric function. The integration has to be performed by using the Landau contour [31]. Note first that in the case  $f(\mathbf{v}) = \rho \delta(\mathbf{v})$ , after an integration by parts, we obtain the explicit relation

$$\omega^2 = (2\pi)^D \hat{u}(k) k^2 \rho. \quad (11)$$

On the other hand, for the Maxwellian distribution function with uniform density

$$f(\mathbf{v}) = \left( \frac{\beta m}{2\pi} \right)^{D/2} \rho e^{-\beta m \frac{v^2}{2}}, \quad (12)$$

the dielectric function can be expressed as

$$\epsilon(\mathbf{k}, \omega) = 1 + (2\pi)^D \hat{u}(k) \beta m \rho W(\sqrt{\beta m} \frac{\omega}{k}) \quad (13)$$

where

$$W(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{x}{x - z} e^{-\frac{x^2}{2}} dx, \quad (14)$$

is the  $W$ -function of plasma physics [31]. Explicitly,

$$W(z) = 1 - z e^{-\frac{z^2}{2}} \int_0^z e^{\frac{y^2}{2}} dy + i \sqrt{\frac{\pi}{2}} z e^{-\frac{z^2}{2}}. \quad (15)$$

The dispersion relation can therefore be written

$$1 + (2\pi)^D \hat{u}(k) \beta m \rho W(\sqrt{\beta m} \frac{\omega}{k}) = 0. \quad (16)$$

We look for solutions of the form  $\omega = i\lambda$  where  $\lambda$  is real. Then, we get

$$\eta(k) = G\left(\sqrt{\frac{\beta m}{2}} \frac{\lambda}{k}\right) \quad (17)$$

where we have set  $\eta(k) = -(2\pi)^D \hat{u}(k) \beta m \rho$  and defined the  $G$ -function

$$G(x) = \frac{1}{1 - \sqrt{\pi} x e^{x^2} \operatorname{erfc}(x)}. \quad (18)$$

For  $x \rightarrow 0$ ,  $G(x) = 1 + \sqrt{\pi}x + \dots$ . For  $x \rightarrow +\infty$ ,  $G(x) = 2x^2(1 + \frac{3}{2x^2} + \dots)$ . For  $x \rightarrow -\infty$ ,  $G(x) \sim -\frac{1}{2\sqrt{\pi}x} e^{-x^2}$ . We note that this function is always positive. Therefore, there exists solutions of the form  $\omega = i\lambda$  with  $\lambda$  real only if  $\hat{u}(k) < 0$ , i.e. for attracting potentials. For neutral plasmas with Maxwellian distribution, the pulsation  $\omega_r$  of the perturbation can never vanish.

The case of neutral stability  $\omega = 0$  corresponds to  $\eta(k) = 1$ . The case of instability ( $\lambda > 0$ ) corresponds to  $\eta(k) > 1$ . In that case, the perturbation grows exponentially as  $\delta f \sim e^{\lambda t}$ . The criterion (17) can be written explicitly

$$1 - \eta(k) \left\{ 1 - \sqrt{\frac{\pi m}{2T}} \frac{\lambda}{k} e^{\frac{m\lambda^2}{2Tk^2}} \operatorname{erfc}\left(\sqrt{\frac{\beta m}{2}} \frac{\lambda}{k}\right) \right\} = 0, \quad (19)$$

where  $\lambda$  is the growth rate. The case of stability ( $\lambda < 0$ ) corresponds to  $\eta(k) < 1$ . In that case, the perturbation decays exponentially as  $\delta f \sim e^{-\gamma t}$ . The damping rate  $\gamma = -\lambda$  is given by

$$\eta(k) = F\left(\sqrt{\frac{\beta m}{2}} \frac{\gamma}{k}\right) \quad (20)$$

where we have defined the  $F$ -function

$$F(x) = \frac{1}{1 + \sqrt{\pi} x e^{x^2} \operatorname{erfc}(-x)}, \quad (21)$$

such that  $F(x) = G(-x)$ . For  $x \rightarrow 0$ ,  $F(x) = 1 - \sqrt{\pi}x + \dots$ . For  $x \rightarrow -\infty$ ,  $F(x) = 2x^2(1 + \frac{3}{2x^2} + \dots)$ . For  $x \rightarrow +\infty$ ,  $F(x) \sim \frac{1}{2\sqrt{\pi}x} e^{-x^2}$ . The criterion (20) can be written explicitly

$$1 - \eta(k) \left\{ 1 + \sqrt{\frac{\pi m}{2T}} \frac{\gamma}{k} e^{\frac{m\gamma^2}{2Tk^2}} \operatorname{erfc}\left(-\sqrt{\frac{\beta m}{2}} \frac{\gamma}{k}\right) \right\} = 0. \quad (22)$$

To summarize, the condition of linear dynamical stability is  $\eta(k) \leq 1$  or, equivalently,

$$1 + (2\pi)^D \hat{u}(k) \beta m \rho \geq 0. \quad (23)$$

It coincides with the criterion of thermodynamical stability (I-95) obtained in Paper I by considering the second order variations of entropy or the possible divergence of the spatial correlation function in Fourier space. The extension of these results to the case of polytropic

(Tsallis) distribution functions, instead of isothermal, is made in [6] for the HMF model and in [23] for self-gravitating systems. More generally, when  $f = f(v)$  depends only on the modulus of the velocity, we have  $\partial f / \partial \mathbf{v} = f'(v) \mathbf{v} / v$  and the condition of linear dynamical stability is

$$1 - (2\pi)^D \hat{u}(k) \int \frac{f'(v)}{v} d^D \mathbf{v} \geq 0. \quad (24)$$

The equality corresponds to the condition of marginal stability  $\omega = 0$  in Eq. (10). We refer to [6, 32] for a more thorough discussion of these results in the case of the HMF model.

#### D. The Euler equation

It is of interest to compare the stability of a system described by the Vlasov equation (7)-(8) to that of a system described by the Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (25)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Phi, \quad (26)$$

with a barotropic equation of state  $p = p(\rho)$  and  $\Phi = \rho * u$ . The energy functional of this barotropic gas is

$$\mathcal{W}[\rho, \mathbf{u}] = \int \rho \int_0^\rho \frac{p(\rho')}{\rho'^2} d\rho' d^D \mathbf{r} + \frac{1}{2} \int \rho \Phi d^D \mathbf{r} + \int \rho \frac{\mathbf{u}^2}{2} d^D \mathbf{r}. \quad (27)$$

This functional is conserved by the Euler equations. Considering the linear dynamical stability of a spatially homogeneous system with respect to the Euler equation, we find that the dispersion relation reads [6]:

$$\omega^2 = c_s^2 k^2 + (2\pi)^D \hat{u}(k) k^2 \rho, \quad (28)$$

where  $c_s^2 = dp/d\rho$  is the velocity of sound. Note that for  $c_s = 0$ , Eq. (28) coincides with Eq. (11). The condition of stability is

$$c_s^2 + (2\pi)^D \hat{u}(k) \rho \geq 0. \quad (29)$$

In the unstable regime,  $\omega$  is imaginary ( $\omega = i\lambda$  with  $\lambda > 0$ ) and the perturbation grows exponentially as  $\delta\rho \sim e^{\lambda t}$ . In the stable regime,  $\omega$  is real and the perturbation oscillates as  $\delta\rho \sim e^{-i\omega t}$ . For an isothermal equation of state  $p = \rho T/m$ , the velocity of sound is  $c_s^2 = T/m$  and the instability criteria (23) and (29) coincide. In fact, this coincidence is general and goes beyond the isothermal distribution as shown in [6, 23] and in the next section. Indeed, for spatially homogeneous systems, the condition of dynamical stability (24) for a kinetic system described by the Vlasov equation with  $f = f(v)$  is the same as the condition of dynamical stability (29) for the corresponding barotropic gas (defined in Sec. II E) described by the Euler equation. Note, however, that the evolution of the perturbation is different in the two systems (kinetic and gaseous) [6].

### E. Nonlinear dynamical stability

It can be shown that a distribution function  $f(\mathbf{r}, \mathbf{v})$  which maximizes a H-function (9) at fixed mass and energy is a nonlinearly dynamically stable stationary solution of the Vlasov equation. This is because  $H$ ,  $E$  and  $M$  are conserved by the Vlasov equation (see [23] for a more detailed discussion and references). Cancelling the first variations of  $H$  at fixed  $E$  and  $M$ , we get

$$C'(f) = -\beta\epsilon - \alpha \quad \Leftrightarrow \quad f = F(\beta\epsilon + \alpha), \quad (30)$$

where  $F(x) = (C')^{-1}(-x)$ . We note that  $f = f(\epsilon)$  depends only on the individual energy  $\epsilon = \frac{v^2}{2} + \Phi(\mathbf{r})$  of the particles and is monotonically decreasing ( $f'(\epsilon) < 0$  according to Eq. (30), assuming  $\beta > 0$ ). This implies that the density  $\rho = \int f d^D \mathbf{v} = \rho(\Phi)$  and the pressure  $p = \frac{1}{D} \int f v^2 d^D \mathbf{v} = p(\Phi)$  are functions of  $\Phi$ . Eliminating  $\Phi$  between these two expressions, we find that the equation of state is barotropic in the sense that  $p = p(\rho)$ . Therefore, to each kinetic system with distribution function  $f = f(\epsilon)$ , we can associate a corresponding barotropic gas with the same equilibrium density distribution. Now, we have  $p'(\Phi) = \frac{1}{D} \int f'(\epsilon) v^2 d^D \mathbf{v} = \frac{1}{D} \int \frac{\partial f}{\partial \mathbf{v}} \cdot \mathbf{v} d^D \mathbf{v} = - \int f d^D \mathbf{v}$  so that  $p'(\Phi) = -\rho(\Phi)$  which corresponds to the condition of hydrostatic equilibrium in a gas  $\nabla p = -\rho \nabla \Phi$ . Then, we get  $p'(\rho) = p'(\Phi)/\rho'(\Phi) = -\rho(\Phi)/\rho'(\Phi)$ . But,  $\rho'(\Phi) = \int f'(\epsilon) d^D \mathbf{v} = \int \frac{\partial f}{\partial v}/v d^D \mathbf{v}$ . Therefore, the velocity of sound  $c_s^2 = p'(\rho)$  can be written  $c_s^2 = -\rho / \int \frac{\partial f}{\partial v}/v d^D \mathbf{v}$ . This relation remains valid if the system is homogeneous [6] so that finally:

$$c_s^2 = \frac{-\rho}{\int \frac{f'(v)}{v} d^D \mathbf{v}}. \quad (31)$$

Substituting this relation in Eq. (24) we see that the instability criteria (24) and (29) coincide as announced.

We now turn to the nonlinear dynamical stability problem. It can be shown that a distribution function  $f(\mathbf{r}, \mathbf{v})$  which minimizes the Casimir-Energy functional  $F = E - TH$  (where  $T = 1/\beta$  is a positive constant) at fixed mass is a nonlinearly dynamically stable stationary solution of the Vlasov equation. Again, this is because  $F$  and  $M$  are conserved by the Vlasov equation. If the distribution function minimizes  $F$  at fixed  $M$  (which is similar to a “canonical” stability criterion in thermodynamics) then it maximizes  $H$  at fixed  $E$  and  $M$  (which is similar to a “microcanonical” stability criterion in thermodynamics) [23]. However, the reciprocal is wrong in general so that the “canonical” criterion  $\{\min F \mid M\}$  is less refined than the “microcanonical” criterion  $\{\max H \mid M, E\}$ , and it just provides a *sufficient* condition of nonlinear dynamical stability. When the two criteria do not coincide, this is similar to a situation of “ensemble inequivalence” in thermodynamics. Such “inequivalence” is observed for the nonlinear dynamical stability of self-gravitating systems such as stellar polytropes and is related to the Antonov first law [22, 23]. However, for spatially homogeneous systems, the two criteria are equivalent in general [6] and we shall use here the simpler “canonical” criterion. To minimize  $F[f]$  at fixed mass, we first minimize  $F[f]$  at fixed density profile  $\rho(\mathbf{r})$ . This yields an optimal distribution  $f_*(\mathbf{r}, \mathbf{v})$  determined by  $C'(f_*) = -\beta\frac{v^2}{2} + \lambda(\mathbf{r})$  where  $\lambda(\mathbf{r})$  can be related to the density, using  $\rho = \int f_* d^D \mathbf{v}$ . Then, we minimize the functional  $F[\rho] = F[f_*]$  at fixed mass. It is shown in [22] that this functional can be written

$$F[\rho] = \int \rho \int_0^\rho \frac{p(\rho')}{\rho'^2} d\rho' d^D \mathbf{r} + \frac{1}{2} \int \rho \Phi d^D \mathbf{r}, \quad (32)$$

where  $p(\rho)$  is the equation of state of the corresponding barotropic gas defined above. As observed in [6, 22, 23], the functional (32) coincides with the energy functional (27) of a barotropic gas described by the Euler equations with  $\mathbf{u} = \mathbf{0}$ . Now, a distribution  $\rho(\mathbf{r})$  which minimizes the energy functional (27) at fixed mass is a nonlinearly dynamically stable stationary solution of the Euler equations (25)-(26). This is because  $\mathcal{W}[\rho, \mathbf{u}]$  is conserved by the Euler equations. Since  $F[\rho]$  and  $\mathcal{W}[\rho, \mathbf{0}]$  coincide, the condition of nonlinear dynamical stability for a homogeneous system described by the Vlasov equation is the same as the condition of nonlinear dynamical stability for the corresponding barotropic gas with respect to the Euler equation. This is the version of the nonlinear Antonov first law for homogeneous systems [6]. The second variations of Eq. (32) are

$$\delta^2 F = \int \frac{p'(\rho)}{2\rho} (\delta\rho)^2 d^D \mathbf{r} + \frac{1}{2} \int \delta\rho \delta\Phi d^D \mathbf{r}, \quad (33)$$

which must be positive for nonlinear dynamical stability. Now, repeating the same steps as in Sec. IV.D of Paper I by simply replacing  $T/m$  by  $c_s^2 = p'(\rho)$ , since  $\rho$  is constant, we find that the condition of nonlinear dynamical stability is

$$c_s^2 + (2\pi)^D \hat{u}(k) \rho \geq 0, \quad (34)$$

for all  $k$ . Using Eq. (31), this can also be expressed as

$$1 - (2\pi)^D \hat{u}(k) \int \frac{f'(v)}{v} d^D \mathbf{v} \geq 0, \quad (35)$$

for all  $k$ . This generalizes the criterion of nonlinear dynamical stability given in [12] for the HMF model. Our approach also gives an interpretation of this criterion in terms of a condition on the velocity of sound in a barotropic gas with the same equation of state as the original kinetic system [6]. Indeed, we must have

$$c_s^2 > (c_s^2)_{\max} \equiv \rho \hat{v}(k)_{\max} \quad (36)$$

for nonlinear dynamical stability. Finally, we note that the criteria (34) and (35) are the same as (29) and (24). This implies that, for homogeneous systems, the conditions of linear and nonlinear dynamical stability coincide.

## F. The Landau equation

The Vlasov equation is very important for systems with long-range interactions because collisional effects become manifest on a timescale of order of  $Nt_D$  or larger. For many realistic systems (e.g., galaxies in astrophysics) the number of particles is so large ( $N \sim 10^{12}$ ) that only the collisionless regime matters for timescales of interest [7]. Therefore, we can consider the limit  $N \rightarrow +\infty$  leading rigorously to the Vlasov equation. However, the limits  $N \rightarrow \infty$  and  $t \rightarrow \infty$  are not interchangeable. For sufficiently long times, collisional effects must be taken into account. This is the case for globular clusters in astrophysics [7] which form smaller groups of stars ( $N \sim 10^6$ ) and whose age is of the order of the relaxation time. We would like now to take the effect of correlations between particles into account in order to describe the “collisional” relaxation due to finite  $N$  effects. In particular, we would like to obtain the form of the collision term to order  $1/N$ . This is the first correction to the Vlasov

regime in the  $N^{-1}$  expansion of the correlation functions. It will describe the dynamics of the system on a timescale  $\sim Nt_D$ .

There are different methods to obtain a kinetic equation for the distribution function  $f(\mathbf{r}, \mathbf{v}, t)$ . One possibility is to close the BBGKY hierarchy by neglecting the cumulant of the three-body correlation function [31], which is of order  $P'_3 \sim N^{-2}$  in the present context. This corresponds to the Kirkwood approximation in plasma physics. Another possibility is to use the Klimontovich approach and develop a quasilinear theory (see, e.g., [3] and Appendix B). A third possibility is to use a projection operator formalism, e.g. [33]. An interest of this approach is that it takes into account non Markovian effects and spatial delocalization. If we neglect collective effects, the projection operator formalism leads to a kinetic equation of the form

$$\begin{aligned} \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \langle \mathbf{F} \rangle \cdot \frac{\partial f}{\partial \mathbf{v}} &= \frac{\partial}{\partial v^\mu} \int_0^t d\tau \int d^D \mathbf{r}_1 d^D \mathbf{v}_1 \mathcal{F}^\mu(1 \rightarrow 0, t) \\ &\times \left\{ \mathcal{F}^\nu(1 \rightarrow 0, t - \tau) \frac{\partial}{\partial v^\nu} + \mathcal{F}^\nu(0 \rightarrow 1, t - \tau) \frac{\partial}{\partial v_1^\nu} \right\} \frac{f}{m}(\mathbf{r}_1, \mathbf{v}_1, t - \tau) f(\mathbf{r}, \mathbf{v}, t - \tau). \end{aligned} \quad (37)$$

Here,  $f(\mathbf{r}, \mathbf{v}, t) = NmP_1(\mathbf{r}, \mathbf{v}, t)$  is the distribution function,  $\langle \mathbf{F} \rangle(\mathbf{r}, t)$  is the (smooth) mean-field force and  $\mathcal{F}^\mu(1 \rightarrow 0, t) = F^\mu(1 \rightarrow 0, t) - \langle F^\mu \rangle(\mathbf{r}, t)$  is the fluctuating force created by particle 1 (with position and velocity  $\mathbf{r}_1, \mathbf{v}_1$ ) on particle 0 (with  $\mathbf{r}, \mathbf{v}$ ) at time  $t$ . Between  $t$  and  $t - \tau$ , the particles are assumed to follow the trajectories determined by the slowly evolving mean-field  $\langle \mathbf{F} \rangle(\mathbf{r}, t)$ . Equation (37) is a non-Markovian integrodifferential equation. We insist on the fact that this equation is valid for an inhomogeneous system while the kinetic equations presented in the sequel will only apply to homogeneous systems. Unfortunately, Eq. (37) remains too complicated for practical purposes and we have to make simplifications. If we consider a spatially homogeneous system for which the distribution function  $f = f(\mathbf{v}, t)$  depends only on the velocity [50], and if we implement a Markovian approximation, the foregoing equation reduces to

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial}{\partial v^\mu} \int_0^{+\infty} d\tau \int d^D \mathbf{r}_1 d^D \mathbf{v}_1 F^\mu(1 \rightarrow 0, t) \\ &\times F^\nu(1 \rightarrow 0, t - \tau) \left( \frac{\partial}{\partial v^\nu} - \frac{\partial}{\partial v_1^\nu} \right) \frac{f}{m}(\mathbf{v}_1, t) f(\mathbf{v}, t), \end{aligned} \quad (38)$$

with  $F^\mu(1 \rightarrow 0, t) = F^\mu(\mathbf{r}_1(t) \rightarrow \mathbf{r}(t))$ . In our approximation, the particles follow linear trajectories with constant velocity since  $\langle \mathbf{F} \rangle = \mathbf{0}$ . Then, the collision term can be simplified (see Appendix A) and we obtain

$$\frac{\partial f}{\partial t} = \pi(2\pi)^D m \frac{\partial}{\partial v^\mu} \int d^D \mathbf{v}_1 d^D \mathbf{k} k^\mu k^\nu \hat{u}(\mathbf{k})^2 \delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}_1)] \left( f_1 \frac{\partial f}{\partial v^\nu} - f \frac{\partial f_1}{\partial v_1^\nu} \right), \quad (39)$$

where  $f = f(\mathbf{v}, t)$  and  $f_1 = f(\mathbf{v}_1, t)$ . Introducing the relative velocity  $\mathbf{u} = \mathbf{v} - \mathbf{v}_1$ , this can be written more conveniently as

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v^\mu} \int d^D \mathbf{v}_1 K^{\mu\nu}(\mathbf{u}) \left( f_1 \frac{\partial f}{\partial v^\nu} - f \frac{\partial f_1}{\partial v_1^\nu} \right), \quad (40)$$

where

$$K^{\mu\nu} = \pi(2\pi)^D m \int d^D \mathbf{k} k^\mu k^\nu \hat{u}(k)^2 \delta(\mathbf{k} \cdot \mathbf{u}). \quad (41)$$

We note that the thermodynamic limit defined in Paper I amounts to considering that the coupling constant scales as  $1/N$ , all other quantities being of order unity. Then, recalling that  $f = NmP_1 \sim N$ , we find that the collision operator in the Landau equation scales as  $1/N$  and represents therefore the first correction to the Vlasov equation. It also results from this observation that the Landau equation will describe the evolution of the system on a timescale of order  $Nt_D$ .

Equation (40) with Eq. (41) is the general form of the Landau equation. Landau derived it for an electronic plasma from the Boltzmann equation in a weak deflexion limit, using a linear trajectory approximation [34]. It can also be obtained from the Fokker-Planck equation by calculating the first and second moments of the velocity increments induced by a succession of two-body encounters [35]. The Landau equation can be further simplified by explicitly evaluating the tensor (41). In  $D = 3$  and  $D = 2$ , we find that

$$K^{\mu\nu} = K_D \left( \delta^{\mu\nu} - \frac{u^\mu u^\nu}{u^2} \right), \quad (42)$$

where

$$K_3 = 8\pi^5 m \frac{1}{u} \int_0^{+\infty} k^3 \hat{u}(k)^2 dk, \quad K_2 = 8\pi^3 m \frac{1}{u} \int_0^{+\infty} k^2 \hat{u}(k)^2 dk. \quad (43)$$

The Landau equation conserves the constants of motion of the Hamiltonian dynamics (mass, energy,...) and increases the Boltzmann entropy (H-theorem). For  $D = 2$  and  $D = 3$ , the distribution function approaches the Maxwellian for  $t \rightarrow +\infty$ . The relaxation time scales as  $t_{relax} \sim Nt_D$ . Therefore, the kinetic theory justifies, at least at order  $1/N$  and for homogeneous systems, the Maxwell distribution predicted by the statistical theory. Since the microcanonical distribution is based on an assumption of ergodicity (see Paper I), it is not granted a priori that the system reaches statistical equilibrium. Thus, the development of a kinetic theory is necessary to validate (or not) the microcanonical distribution. For inhomogeneous systems, it is not clear whether the kinetic equation (37) truly relaxes towards the mean-field Maxwell-Boltzmann distribution due to non-Markovian effects and spatial delocalization. Therefore, the validity of the Boltzmann distribution depends on the dynamics and a kinetic theory is required to justify its relevance.

The collisional evolution of stellar systems (that are inhomogeneous) is usually described by the Vlasov-Landau equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \langle \mathbf{F} \rangle \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial v^\mu} \int d^D \mathbf{v}_1 K^{\mu\nu}(\mathbf{u}) \left( f_1 \frac{\partial f}{\partial v^\nu} - f \frac{\partial f_1}{\partial v_1^\nu} \right), \quad (44)$$

with now  $f = f(\mathbf{r}, \mathbf{v}, t)$  and  $f_1 = f(\mathbf{r}, \mathbf{v}_1, t)$ . This equation, obtained by combining Eqs. (7) and (40), assumes that the collisions can be treated as local (see [33] for a critical discussion of this approximation). It has to be coupled to the Poisson equation. For the gravitational potential in  $D = 3$ ,

$$K_3 = 2\pi m G^2 \frac{1}{u} \int_0^{+\infty} \frac{dk}{k}. \quad (45)$$

This quantity exhibits a well-known logarithmic divergence at small and large scales. The divergence at small scales is common to both plasmas and gravitational systems. It is due

to the failure of the linear trajectory approximation. It is regularized by cutting the integral at the Landau length, corresponding to the impact parameter for large angle collisions. The divergence at large scales is specific to the gravitational case (in the plasma case, the integral must be cut-off at the Debye length as shown by the Lenard-Balescu treatment). It is due to the long-range nature of gravity and the absence of shielding. This problem has been the subject of several studies. In their stochastic analysis, Chandrasekhar & von Neumann [36] argue that the integral has to be cut-off at the interparticle distance since the distribution of the gravitational field (a Lévy distribution) is dominated by the contribution of the nearest neighbor. Alternatively, most astrophysicists argue that the integral has to be cut-off at the size of the system, of the order of the Jeans length, which is presumably the gravitational analogue of the Debye length. In any case, because of the logarithmic divergence, the relaxation time scales as  $t_{relax} \sim (N/\ln N)t_D$  instead of  $Nt_D$ . For the gravitational or Coulombian potential in  $D = 2$ , the integral behaves as  $\int_0^{+\infty} dk/k^2$  and it diverges linearly for  $\lambda = 2\pi/k \rightarrow +\infty$ . This suggests that  $K_2$  is proportional to the large-scale cut-off (the Debye length in plasma physics). However, a more precise study based on the Lenard-Balescu equation taking into account collective effects is required to ascertain this result.

In  $D = 1$ , the Landau equation becomes

$$\frac{\partial f}{\partial t} = K \frac{\partial}{\partial v} \int dv_1 \delta(v - v_1) \left( f_1 \frac{\partial f}{\partial v} - f \frac{\partial f_1}{\partial v_1} \right), \quad (46)$$

where

$$K = 4\pi^2 m \int_0^{+\infty} k \hat{u}(k)^2 dk. \quad (47)$$

Due to the  $\delta$ -function, the collision term vanishes identically. This implies that the distribution function does not evolve on a timescale of order  $Nt_D$ , i.e.  $\partial f/\partial t = 0$ . Since the Landau equation is valid at order  $1/N$ , the relaxation towards statistical equilibrium will be due to non trivial correlations which appear at higher order in the large  $N$  expansion. This implies that, in 1D, the relaxation time is longer than  $Nt_D$ . For the HMF model, Yamaguchi *et al.* [12] report a non trivial scaling  $N^{1.7}$ . A similar cancellation of the collision term at order  $1/N$  occurs for 2D point vortices when the profile of angular velocity is monotonic. Indeed, neglecting collective effects, the kinetic equation describing the collisional evolution of an axisymmetric point vortex system is [9]:

$$\frac{\partial P}{\partial t} = -\frac{N\gamma^2}{4r} \frac{\partial}{\partial r} \int_0^{+\infty} r_1 dr_1 \delta(\Omega - \Omega_1) \ln \left[ 1 - \left( \frac{r_{<}}{r_{>}} \right)^2 \right] \left( \frac{1}{r} P_1 \frac{\partial P}{\partial r} - \frac{1}{r_1} P \frac{\partial P_1}{\partial r_1} \right), \quad (48)$$

where  $P = P(r, t)$ ,  $P_1 = P(r_1, t)$  and  $r_{<}$  (resp.  $r_{>}$ ) is the min (resp. max) of  $r$  and  $r_1$ . The evolution is due to distant collisions. The collision operator is due to a resonance between vortices rotating with equal angular velocity  $\Omega(r) = \Omega(r_1)$  and vanishes identically when there is no resonance. Equation (48), derived in Chavanis [9] by using projection operator technics, is the vortex analogue of the Landau equation (40) for plasmas. A more general kinetic equation, taking into account collective effects, has been derived by Dubin & O'Neil [8] from the Klimontovich approach. It represents the analogue of the Lenard-Balescu equation (49) in plasma physics. As discussed above, the relaxation of 2D point vortices and of the HMF model toward statistical equilibrium is not clearly understood and demands to go to higher order in the  $1/N$  expansion.

### G. The Lenard-Balescu equation

The Lenard-Balescu equation can be obtained from the first two equations of the BBGKY hierarchy by neglecting non-trivial three-body correlations and assuming that the two-particle correlation function relaxes much faster than the one-particle distribution function (Bogoliubov's hypothesis) [31]. It can also be obtained from the Klimontovich equation by developing a quasilinear theory (see Appendix B). For a homogeneous system, the Lenard-Balescu equation can be written

$$\frac{\partial f}{\partial t} = \pi(2\pi)^D m \frac{\partial}{\partial v^\mu} \int d^D \mathbf{v}_1 d^D \mathbf{k} k^\mu k^\nu \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}_1)] \left( f_1 \frac{\partial f}{\partial v^\nu} - f \frac{\partial f_1}{\partial v_1^\nu} \right) \quad (49)$$

where

$$\epsilon(\mathbf{k}, \omega) = 1 + (2\pi)^D \hat{u}(\mathbf{k}) \int \frac{\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}}}{\omega - \mathbf{k} \cdot \mathbf{v}} d^D \mathbf{v}, \quad (50)$$

is the dielectric function. The Lenard-Balescu equation can be seen as a generalization of the Landau equation taking into account collective effects. The classical Landau equation is recovered in the limit  $|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2 \simeq 1$ . In D=1, the Lenard-Balescu equation becomes

$$\frac{\partial f}{\partial t} = 2\pi^2 m \frac{\partial}{\partial v} \int dv_1 |k| dk \frac{\hat{u}(k)^2}{|\epsilon(k, kv)|^2} \delta(v - v_1) \left( f_1 \frac{\partial f}{\partial v} - f \frac{\partial f_1}{\partial v_1} \right) \quad (51)$$

where

$$\epsilon(k, \omega) = 1 + 2\pi \hat{u}(k) \int \frac{f'(v)}{\omega/k - v} dv. \quad (52)$$

We see that the collision term again vanishes identically at the order  $1/N$ .

### H. Test particle in a thermal bath: the Fokker-Planck equation

The Lenard-Balescu equation can also be used to describe the evolution of a test particle in a bath of field particles at equilibrium. In that case, we have to consider that the distribution  $f_1$  of the bath is *given*, i.e.  $f_1(\mathbf{v}, t) = f_0(\mathbf{v})$  where  $f_0(\mathbf{v})$  is a stable stationary solution of the Vlasov equation (bath distribution). This procedure transforms the integro-differential equation (49) into a differential equation for the density probability  $P(\mathbf{v}, t)$  of finding the test particle with velocity  $\mathbf{v}$  at time  $t$ . It reads

$$\frac{\partial P}{\partial t} = \pi(2\pi)^D m \frac{\partial}{\partial v^\mu} \int d^D \mathbf{v}_1 d^D \mathbf{k} k^\mu k^\nu \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}_1)] \left( \frac{\partial}{\partial v^\nu} - \frac{\partial}{\partial v_1^\nu} \right) f_0(\mathbf{v}_1) P(\mathbf{v}, t), \quad (53)$$

where

$$\epsilon(\mathbf{k}, \omega) = 1 + (2\pi)^D \hat{u}(\mathbf{k}) \int \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k} \cdot \mathbf{v}} d^D \mathbf{v}. \quad (54)$$

Equation (53) can be written in the form of a Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v^\mu} \left( D^{\mu\nu} \frac{\partial P}{\partial v^\nu} + P \eta^\mu \right), \quad (55)$$

which involves a diffusion term

$$D^{\mu\nu} = \pi(2\pi)^D m \int d^D \mathbf{v}_1 d^D \mathbf{k} k^\mu k^\nu \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}_1)] f_0(\mathbf{v}_1) \quad (56)$$

and a friction term

$$\eta^\mu = -\pi(2\pi)^D m \int d^D \mathbf{v}_1 d^D \mathbf{k} k^\mu k^\nu \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} \delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}_1)] \frac{\partial f_0}{\partial v_1^\nu}(\mathbf{v}_1). \quad (57)$$

The diffusion coefficient is due to the fluctuations of the force and is given by the Kubo formula

$$D^{\mu\nu} = \int_0^{+\infty} \langle F^\mu(0) F^\nu(t) \rangle dt. \quad (58)$$

The dynamical friction results from a polarization process and can be explicitly calculated by developing a linear response theory. In the present context, the coefficients of diffusion and friction depend on the velocity. Hence, it is more proper to write Eq. (55) in a form which is fully consistent with the general Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial v^\mu \partial v^\nu} \left( P \frac{\langle \Delta v^\mu \Delta v^\nu \rangle}{\Delta t} \right) - \frac{\partial}{\partial v^\mu} \left( P \frac{\langle \Delta v^\mu \rangle}{\Delta t} \right), \quad (59)$$

with

$$\frac{\langle \Delta v^\mu \Delta v^\nu \rangle}{\Delta t} = 2D^{\mu\nu}, \quad \frac{\langle \Delta v^\mu \rangle}{\Delta t} = \frac{\partial D^{\mu\nu}}{\partial v^\nu} - \eta^\nu = -2\eta^\nu, \quad (60)$$

where the last equality is obtained from Eqs. (56) and (57) by using an integration by parts. We refer to Ichimaru [31] for a more comprehensive discussion of the test particle approach and for the connection between the Lenard-Balescu equation and the Fokker-Planck equation in the thermal bath approximation. Our aim, here, is to give explicit expressions of the Fokker-Planck equation and diffusion coefficient in particular cases.

### 1. The isothermal case

For the Boltzmann distribution

$$f_0(\mathbf{v}_1) = \left( \frac{\beta m}{2\pi} \right)^{D/2} \rho e^{-\beta m \frac{v_1^2}{2}}, \quad (61)$$

corresponding to statistical equilibrium (thermal bath), we easily find from Eqs. (56) and (57) that  $\eta^\mu = \beta m D^{\mu\nu} v^\nu$ . Therefore, the friction coefficient is given by a generalized Einstein relation and the Fokker-Planck equation (55) takes the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v^\mu} \left[ D^{\mu\nu}(\mathbf{v}) \left( \frac{\partial P}{\partial v^\nu} + \beta m P v^\nu \right) \right]. \quad (62)$$

This is similar to the Kramers equation in Brownian theory. However, in the present context, the diffusion coefficient  $D^{\mu\nu}$  is anisotropic and depends on the velocity  $\mathbf{v}$  of the test particle. We note that for  $t \rightarrow +\infty$ , the velocity distribution of the test particle  $P(\mathbf{v}, t)$  relaxes to the Maxwellian distribution (61) of the bath (thermalization). Writing

$$\delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}_1)] = \int e^{i\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}_1)t} \frac{dt}{2\pi}, \quad (63)$$

we can put the diffusion coefficient (56) in the form

$$D^{\mu\nu}(\mathbf{v}) = (2\pi)^{2D} m \int_0^{+\infty} dt \int d^D \mathbf{k} k^\mu k^\nu \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} e^{i\mathbf{k} \cdot \mathbf{v}t} \hat{f}_0(\mathbf{k}t). \quad (64)$$

It will be shown in Sec. III that this equation can be interpreted as a Kubo formula (58) where  $t$  plays the role of time. For a Maxwellian distribution, the Fourier transform  $\hat{f}_0(\mathbf{k}t)$  is a Gaussian. We can thus easily integrate on time  $t$  to obtain

$$D^{\mu\nu}(\mathbf{v}) = \pi(2\pi)^D m \rho \left( \frac{\beta m}{2\pi} \right)^{1/2} \int d^D \mathbf{k} \frac{k^\mu k^\nu}{k} \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2} e^{-\beta m \frac{(\mathbf{k} \cdot \mathbf{v})^2}{2k^2}}. \quad (65)$$

On the other hand, for a Maxwellian distribution, the dielectric function is given by Eq. (13). Using (15), we can thus rewrite the Fokker-Planck equation in the form

$$\frac{\partial P}{\partial t} = \frac{1}{t_R} \frac{\partial}{\partial x^\mu} \left[ G^{\mu\nu}(\mathbf{x}) \left( \frac{\partial P}{\partial x^\nu} + 2P x^\nu \right) \right], \quad (66)$$

where

$$G^{\mu\nu}(\mathbf{x}) = L^{D+1} \int d^D \mathbf{k} \frac{\hat{k}^\mu \hat{k}^\nu k \eta(k)^2 e^{-(\hat{\mathbf{k}} \cdot \mathbf{x})^2}}{[1 - \eta(k) B(\hat{\mathbf{k}} \cdot \mathbf{x})]^2 + \pi \eta(k)^2 (\hat{\mathbf{k}} \cdot \mathbf{x})^2 e^{-2(\hat{\mathbf{k}} \cdot \mathbf{x})^2}} \quad (67)$$

and  $t_R$  is a relaxation timescale defined by

$$t_R^{-1} = \frac{1}{(2\pi)^D} \left( \frac{\pi}{8D} \right)^{1/2} \frac{v_m}{n} \frac{1}{L^{D+1}}. \quad (68)$$

In the foregoing formulae, we have set  $\mathbf{x} = (\beta m/2)^{1/2} \mathbf{v}$ ,  $\hat{\mathbf{k}} = \mathbf{k}/k$ ,  $B(x) = 1 - 2xe^{-x^2} \int_0^x e^{y^2} dy$  and  $\eta(k) = -(2\pi)^D \hat{u}(k) \beta m \rho$ . In addition,  $v_m = (D/\beta m)^{1/2}$  is the r.m.s. velocity and  $L$  is a lengthscale (size of the domain) introduced to make Eq. (67) dimensionless. The function  $B(x)$  can be written  $B(x) = 1 - 2xD(x)$  where  $D(x) = e^{-x^2} \int_0^x e^{y^2} dy$  is Dawson's integral. We note the asymptotic behaviors  $B(x) = 1 - 2x^2 + \dots$  for  $x \rightarrow 0$  and  $B(x) \sim -\frac{1}{2x^2}$  for  $x \rightarrow +\infty$ . Equations (66)-(68) provide the general form of the Fokker-Planck equation describing the evolution of a test particle in a thermal bath. They generalize the results obtained in [6, 10, 11] for the HMF model where the potential of interaction is truncated to one single mode and  $D = 1$ . We note that the relaxation time scales as  $t_R \sim N t_D$  where we have introduced the dynamical time  $t_D \sim L/v_m$  and the particle number  $N \sim n L^D$ . As indicated previously, the Fokker-Planck collision term scales as  $1/N$  in the thermodynamic limit defined in Paper I.

We also note that in the case of the Coulombian interaction for which  $\eta(k) = -k_D^2/k^2$  where  $k_D$  is the Debye wavenumber, the integral over  $k$  in Eq. (67) is of the form

$$I_D = \int_0^{+\infty} \frac{k^D dk}{(B + k^2/k_D^2)^2 + C^2} \quad (69)$$

where  $B \equiv B(\hat{\mathbf{k}} \cdot \mathbf{x})$  and  $C \equiv \sqrt{\pi} |\hat{\mathbf{k}} \cdot \mathbf{x}| e^{-(\hat{\mathbf{k}} \cdot \mathbf{x})^2}$  are independent of  $k$ . The integral (69) can be rewritten  $I_D = k_D^{D+1} C^{\frac{D-3}{2}} \Phi_D(B/C)$  with  $\Phi_D(x) = \int_0^{+\infty} t^D dt / ((x+t^2)^2 + 1)$ . In the Landau approximation, we have instead  $I_D^{Landau} = k_D^4 \int_0^{+\infty} k^{D-4} dk$  which presents the divergences mentioned previously. In  $D = 3$ , the integral (69) is now convergent for  $k \rightarrow 0$  and, using the dominant approximation of Chandrasekhar [31], we have  $I_3 \sim k_D^4 \ln(k_m/k_D)$  where  $k_m$  is a small scale cut-off. Therefore, the Lenard-Balescu treatment justifies to cut the logarithmic divergence at large scales at the Debye length  $k_D^{-1}$  [31]. In  $D = 2$ , we note that the integral (69) is convergent for all values of  $k$ . The evaluation of the diffusion coefficient (67) in that case is left for a future study. Finally, the case  $D = 1$  will be treated in Sec. II H 3.

## 2. The Landau approximation

If we make the Landau approximation  $|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2 \simeq 1$  which amounts to neglecting collective effects, we can rewrite the Fokker-Planck equation in the form

$$\frac{\partial P}{\partial t} = \frac{1}{t_R} \frac{\partial}{\partial x^\mu} \left[ G^{\mu\nu}(\mathbf{x}) \left( \frac{\partial P}{\partial x^\nu} + 2P x^\nu \right) \right], \quad (70)$$

where now

$$G^{\mu\nu}(\mathbf{x}) = \int d^D \hat{\mathbf{k}} \hat{k}^\mu \hat{k}^\nu e^{-(\hat{\mathbf{k}} \cdot \mathbf{x})^2}, \quad (71)$$

$$t_R^{-1} = \left(\frac{\pi}{8}\right)^{1/2} D^{3/2} (2\pi)^D \frac{\rho m}{v_m^3} \int_0^{+\infty} k^D \hat{u}(k)^2 dk. \quad (72)$$

Equation (71) also represents the asymptotic behavior of the diffusion coefficient (67) for  $|\mathbf{v}| \rightarrow +\infty$  and for  $T \rightarrow +\infty$  since in these limits  $\epsilon \simeq 1$ . In that case, the tensor  $G^{\mu\nu}$  given by Eq. (71) can be calculated explicitly by introducing spherical or polar systems of coordinates. In  $D = 3$ ,

$$G^{\mu\nu} = (G_{\parallel} - \frac{1}{2}G_{\perp}) \frac{x^\mu x^\nu}{x^2} + \frac{1}{2}G_{\perp} \delta^{\mu\nu}, \quad (73)$$

with

$$G_{\parallel} = \frac{2\pi^{3/2}}{x} G(x), \quad G_{\perp} = \frac{2\pi^{3/2}}{x} [\text{erf}(x) - G(x)], \quad (74)$$

where

$$G(x) = \frac{2}{\sqrt{\pi}} \frac{1}{x^2} \int_0^x t^2 e^{-t^2} dt = \frac{1}{2x^2} \left[ \text{erf}(x) - \frac{2x}{\sqrt{\pi}} e^{-x^2} \right], \quad (75)$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \quad (76)$$

In  $D = 2$ ,

$$G^{\mu\nu} = (G_{\parallel} - G_{\perp}) \frac{x^\mu x^\nu}{x^2} + G_{\perp} \delta^{\mu\nu}, \quad (77)$$

with

$$G_{\parallel} = \pi e^{-\frac{x^2}{2}} \left[ I_0\left(\frac{x^2}{2}\right) - I_1\left(\frac{x^2}{2}\right) \right], \quad G_{\perp} = \pi e^{-\frac{x^2}{2}} \left[ I_0\left(\frac{x^2}{2}\right) + I_1\left(\frac{x^2}{2}\right) \right]. \quad (78)$$

In the above expressions  $G_{\parallel}$  and  $G_{\perp}$  are the diffusion coefficients in the directions parallel and perpendicular to the velocity of the test particle. Finally, in  $D = 1$ ,

$$G(x) = 2e^{-x^2}. \quad (79)$$

We can also develop a test particle approach in the case of 2D point vortices to describes the stochastic evolution of a test vortex in a bath of field vortices at statistical equilibrium [4, 9, 37]. Making a thermal bath approximation, we can transform the kinetic equation (48) into a Fokker-Planck equation of the form

$$\frac{\partial P}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ r D(r) \left( \frac{\partial P}{\partial r} + \beta \gamma P \frac{d\psi_{eq}}{dr} \right) \right], \quad (80)$$

with a diffusion coefficient

$$D(r) = \frac{\gamma}{8} \frac{1}{|\Sigma(r)|} \ln N \langle \omega \rangle_{eq}(r), \quad (81)$$

where  $\Sigma(r) = r \Omega'_{eq}(r)$  is the local shear created by the statistical distribution of field vortices (the angular velocity  $\Omega_{eq}(r)$  is related to the vorticity by  $\omega_{eq}(r) = (1/r)(r^2 \Omega_{eq})'$ ). In Eq. (80),  $P(r, t)$  denotes the density probability of finding the test vortex in  $r$  at time  $t$ . In the more general case where the field vortices have a distribution  $\omega_0(r)$  which is a stable stationary solution of the 2D Euler equation (not necessarily the equilibrium Boltzmann distribution), the Fokker-Planck equation (80) is replaced by

$$\frac{\partial P}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ r D(r) \left( \frac{\partial P}{\partial r} - P \frac{d \ln \omega_0}{dr} \right) \right], \quad (82)$$

and the diffusion coefficient is still given by Eq. (81) with  $\omega_0$  in place of  $\langle \omega \rangle_{eq}$ . In particular, for a vorticity profile  $\omega_0(r) = A e^{-\lambda r^2}$  of the field vortices, it is easy to see that the diffusion coefficient of the test vortex decreases like  $D(r) \sim r^2 e^{-\lambda r^2}$  for  $r \rightarrow +\infty$ . We emphasize the analogies with the Fokker-Planck equations derived previously in the case of material particles. Note that for point vortices, the dynamical friction is replaced by a systematic drift [37] along the vorticity gradient.

### 3. The case $D = 1$

In  $D = 1$ , the Fokker-Planck equation (53) can be written

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v} \left[ D(v) \left( \frac{\partial P}{\partial v} - P \frac{d \ln f_0}{dv} \right) \right], \quad (83)$$

where  $D(v)$  is given by

$$D(v) = 4\pi^2 m f_0(v) \int_0^{+\infty} dk \frac{k \hat{u}(k)^2}{|\epsilon(k, kv)|^2}. \quad (84)$$

Equation (83) is similar to a Fokker-Planck equation describing the motion of a Brownian particle in a potential  $U(\mathbf{v}) = -\ln f_0(\mathbf{v})$  created by the field particles. In  $D = 1$ , the distribution function of the test particle  $P(v, t)$  always relaxes toward the distribution of the bath  $f_0(v)$  for  $t \rightarrow +\infty$  (the typical timescale governing the approach of the test particle to the bath distribution is  $\sim Nt_D$ ). In other dimensions, we see from Eq. (53) that this is true only when  $f_0(\mathbf{v})$  is the Maxwellian distribution (statistical equilibrium state). The particularity of the dimension  $D = 1$  is that the Lenard-Balescu collision operator (49) cancels out for *any* distribution function  $f(v)$  while in  $D = 2$  and  $D = 3$  the cancellation of the collision operator occurs only when the distribution is Maxwellian. In  $D = 2$  and  $D = 3$ , an arbitrary distribution of the field particles  $f_0(\mathbf{v}, t)$  changes on a timescale of order  $Nt_D$  as it relaxes to the Maxwellian due to the development of correlations (finite  $N$  effects). Therefore, the test particle approach is valid only for  $t \ll Nt_D$ , in an interval of times where we can consider that  $f_0(\mathbf{v}, t)$  is approximately stationary. But in that case, the distribution of the test particle  $P(\mathbf{v}, t)$  does not relax toward  $f_0(\mathbf{v})$ . It is only when  $f_0(\mathbf{v})$  is the Maxwellian that we can consider that the distribution of the field particles is frozen (for any time). In that case,  $P(\mathbf{v}, t)$  relaxes toward  $f_0(\mathbf{v})$  in a time  $\sim Nt_D$ . The situation is different in  $D = 1$ . In  $D = 1$ , a stable stationary solution of the Vlasov equation (any) does *not* change on a timescale of order  $Nt_D$ . Therefore, we can consider that the distribution of the field particles  $f_0(v)$  is frozen on a time  $\sim Nt_D$  (or larger) which is precisely the time it takes to a test particle to relax toward  $f_0(v)$ . The fact that the distribution  $P(v, t)$  of a test particle relaxes to any distribution  $f_0(v)$  of the field particles explains why the distribution of the field particles does not change on a time  $\sim Nt_D$ .

For the Maxwellian distribution in  $D = 1$ , one has

$$\frac{\partial P}{\partial t} = \frac{1}{t_R} \frac{\partial}{\partial x} \left[ G(x) \left( \frac{\partial P}{\partial x} + 2Px \right) \right], \quad (85)$$

with

$$G(x) = 2L^2 \int_0^{+\infty} dk \frac{k\eta(k)^2 e^{-x^2}}{[1 - \eta(k)B(x)]^2 + \pi\eta(k)^2 x^2 e^{-2x^2}}, \quad (86)$$

$$t_R^{-1} = \frac{1}{2} \left( \frac{1}{8\pi} \right)^{1/2} \frac{v_m}{n} \frac{1}{L^2}. \quad (87)$$

These results extend those obtained in [6, 10, 11] for the HMF model for which

$$G(x) = \frac{2L^2 e^{-x^2}}{(T/T_c - B(x))^2 + \pi x^2 e^{-2x^2}}. \quad (88)$$

Note that for periodic potentials, the integral over  $\mathbf{k}$  must be replaced by a discrete summation over the different modes. On the other hand, for a Coulombian potential  $\eta(k) = -k_D^2/k^2$ , the integration on  $k$  can be made explicitly, using  $\Phi_1(x) = \pi/4 - \tan^{-1}(x)/2$  and we obtain

$$G(x) = \frac{\sqrt{\pi} L^2 k_D^2}{2|x|} \left[ 1 - \frac{2}{\pi} \tan^{-1} \left( \frac{B(x)}{\sqrt{\pi}|x|e^{-x^2}} \right) \right], \quad (89)$$

with asymptotic behaviors  $G(x) \sim \sqrt{\pi} L^2 k_D^2 / |x|$  for  $x \rightarrow \pm\infty$  and  $G(0) = L^2 k_D^2$ .

## I. Temporal correlation functions

We now wish to determine the temporal correlation function of the force experienced by the test particle. If we ignore collective effects, the temporal correlation function is given by

$$\langle F^\mu(0)F^\nu(t) \rangle = N \int F^\mu(1 \rightarrow 0, 0)F^\nu(1 \rightarrow 0, t)P_1(\mathbf{r}_1, \mathbf{v}_1)d^D\mathbf{r}_1d^D\mathbf{v}_1. \quad (90)$$

Using Fourier transforms and making a linear trajectory approximation as in Appendix A, we obtain

$$\langle F^\mu(0)F^\nu(t) \rangle = m(2\pi)^D \int k^\mu k^\nu \hat{u}(\mathbf{k})^2 e^{-i\mathbf{k} \cdot \mathbf{v}t} f_0(\mathbf{v}_1) d^D\mathbf{v}_1 d^D\mathbf{k}. \quad (91)$$

We can also write Eq. (91) in the form

$$\langle F^\mu(0)F^\nu(t) \rangle = m(2\pi)^{2D} \int k^\mu k^\nu \hat{u}(\mathbf{k})^2 e^{-i\mathbf{k} \cdot \mathbf{v}t} \hat{f}_0(\mathbf{k}t) d^D\mathbf{k}. \quad (92)$$

For a Maxwellian distribution, we get

$$\langle F^\mu(0)F^\nu(t) \rangle = \rho m(2\pi)^D \int k^\mu k^\nu \hat{u}(\mathbf{k})^2 e^{-i\mathbf{k} \cdot \mathbf{v}t} e^{-\frac{k^2 t^2}{2\beta m}} d^D\mathbf{k}. \quad (93)$$

For the gravitational interaction, we can easily perform the integrations by introducing a spherical system of coordinates. The correlation function can finally be written

$$C^{\mu\nu} = (C_\parallel - \frac{1}{2}C_\perp) \frac{v^\mu v^\nu}{v^2} + \frac{1}{2}C_\perp \delta^{\mu\nu}, \quad (94)$$

with

$$C_\parallel = \frac{4\pi\rho m G^2}{vt} G(x), \quad (95)$$

$$C_\perp = \frac{4\pi\rho m G^2}{vt} [\text{erf}(x) - G(x)]. \quad (96)$$

where  $\mathbf{x} = (\beta m/2)^{1/2} \mathbf{v}$  and the function  $G(x)$  is defined by Eq. (75). We note the result

$$\langle \mathbf{F}(0) \cdot \mathbf{F}(t) \rangle = \frac{4\pi\rho m G^2}{vt} \text{erf}(x), \quad (97)$$

which shows that the correlation function of the gravitational force decreases as  $t^{-1}$ . This asymptotic result was first noted by Chandrasekhar [38] using a different approach. The diffusion coefficient is then obtained from the Kubo formula (58) by integrating over time  $t$ . This returns Eqs. (73)-(75) except that the logarithmic divergence  $\int dk/k$  in Eq. (72) now appears on the time integration  $\int dt/t$ . The divergence at large scales is connected to the very slow (algebraic) decay of the temporal correlation function. This algebraic decay is strikingly in contrast with usual Markov processes in which the correlations decrease exponentially rapidly with time. Coming back to Eq. (93), we note that each mode has a

gaussian decay  $\sim \exp[-k^2 t^2 / 2\beta m]$  but the integration over all modes leads to an algebraic behavior  $\sim t^{-1}$ .

If we come back to the general problem and take into account collective effects, it is shown in Appendix B that

$$\langle F^\mu(0)F^\nu(t) \rangle = m(2\pi)^D \int k^\mu k^\nu \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_1)|^2} e^{-i\mathbf{k} \cdot \mathbf{u}t} f_0(\mathbf{v}_1) d^D \mathbf{v}_1 d^D \mathbf{k}. \quad (98)$$

If we integrate over time  $t$  and use the Kubo formula (58), we recover the expression (56) of the diffusion coefficient. Introducing

$$Q(\mathbf{k}, t) = \int e^{i\mathbf{k} \cdot \mathbf{v}_1 t} \frac{f_0(\mathbf{v}_1)}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}_1)|^2} d^D \mathbf{v}_1, \quad (99)$$

we can write the correlation function more compactly as

$$\langle F^\mu(0)F^\nu(t) \rangle = m(2\pi)^D \int k^\mu k^\nu \hat{u}(\mathbf{k})^2 e^{-i\mathbf{k} \cdot \mathbf{v}t} Q(\mathbf{k}, t) d^D \mathbf{k}. \quad (100)$$

For a Maxwellian distribution (61),

$$Q(\mathbf{k}, t) = \rho \left( \frac{\beta m}{2\pi} \right)^{1/2} \frac{1}{k} \int_{-\infty}^{+\infty} e^{i\omega t} \frac{e^{-\beta m \frac{\omega^2}{2k^2}}}{|\epsilon(k, \omega)|^2} d\omega, \quad (101)$$

where we have chosen  $\mathbf{k}$  in the direction of  $x$  and set  $\omega = kv_x$ . To determine the large time asymptotics of Eq. (101), we need to investigate the poles of the function

$$f(\omega) = \frac{e^{-\beta m \frac{\omega^2}{2k^2}}}{|\epsilon|^2(k, \omega)}. \quad (102)$$

Using the notations introduced in Eq. (67), we have

$$|\epsilon|^2(k, \omega) = \left[ 1 - \eta(k) B \left( \sqrt{\frac{\beta m}{2}} \frac{\omega}{k} \right) \right]^2 + \eta(k)^2 \frac{\pi}{2} \beta m \left( \frac{\omega}{k} \right)^2 e^{-\frac{\beta m \omega^2}{2k^2}}. \quad (103)$$

Setting  $\omega = i\lambda$  where  $\lambda$  is real, we find after straightforward algebra that the foregoing expression can be rewritten

$$|\epsilon|^2(k, i\lambda) = \epsilon(k, i\lambda) \epsilon(k, -i\lambda), \quad (104)$$

where we recall that

$$\epsilon(k, i\lambda) = 1 - \eta(k)/G \left( \sqrt{\frac{\beta m}{2}} \frac{\lambda}{k} \right). \quad (105)$$

Equation (102) can thus be rewritten

$$f(\omega) = \frac{e^{\beta m \frac{\lambda^2}{2k^2}}}{\epsilon(k, i\lambda) \epsilon(k, -i\lambda)}. \quad (106)$$

Clearly, this is an even function of  $\lambda$ . We need to determine the values of  $\lambda$  for which the denominator vanishes. Since the distribution of the bath is stable by hypothesis ( $T > T_c$ ), we have  $\eta(k) < 1$  for all  $k$ . Therefore, the denominator cancels out for  $\lambda = \pm\gamma$  where  $\gamma > 0$  is determined by  $\epsilon(k, -i\gamma) = 0$ . Thus  $\gamma$  coincides with the damping rate of the stable perturbed solutions of the Vlasov equation (see Sec. II C). It is given as a function of  $k$  by Eq. (20). Next, we consider  $\lambda = \pm\gamma + \Delta$  where  $\Delta \ll 1$ . Expanding Eq. (106) for small values of  $\Delta$ , we find after elementary calculations that for  $\omega \rightarrow \pm i\gamma$

$$f(\omega) \sim \frac{K(\gamma)}{\omega^2 + \gamma^2}, \quad (107)$$

where the constant is given by

$$K(\gamma) = \frac{2}{\sqrt{\pi}} \frac{k^2}{\beta m} \frac{1}{|F'(\sqrt{\frac{\beta m}{2}} \frac{\gamma}{k})|}. \quad (108)$$

Since  $f(\omega)$  behaves like a Lorentzian for  $\omega \rightarrow \pm i\gamma$ , this implies that, for  $t \rightarrow +\infty$ , the different modes in the correlation function decrease exponentially rapidly as

$$Q(\mathbf{k}, t) \sim \rho \sqrt{\frac{2}{\beta m}} \frac{k}{\gamma_k} \frac{1}{|F'(\sqrt{\frac{\beta m}{2}} \frac{\gamma_k}{k})|} e^{-\gamma_k t}, \quad (109)$$

with a rate

$$\gamma_k = \sqrt{\frac{2}{\beta m}} k F^{-1}[\eta(k)] \quad (110)$$

depending on the wave vector  $k$  (see Eq. (20)). These results generalize those obtained in [6, 10] for the HMF model. In this case, the correlation function decreases like

$$\langle F(0)F(t) \rangle \sim \frac{k^2 M}{8\pi^2} \frac{\sqrt{2T}}{\gamma} \frac{1}{|F'(\frac{\gamma}{\sqrt{2T}})|} \cos(vt) e^{-\gamma t}, \quad (111)$$

where the decay rate

$$\gamma = (2/\beta)^{1/2} F^{-1}(\eta) \quad (112)$$

only depends on the temperature (recall that  $\eta = \beta k M / 4\pi$ ). For  $T \rightarrow T_c^+$ ,  $\gamma \sim (8/kM)^{1/2} (T - T_c)$  and for  $T \rightarrow +\infty$ ,  $\gamma \sim \sqrt{2T \ln T}$  (to leading order). Close to the critical temperature, the correlation function decreases very slowly. *Note that this slow decay may invalidate the Markovian approximation close to the critical point and lead to dynamical anomalies.* On the other hand, at high temperatures, the decay is very fast. In fact, had we ignored collective effects and used Eq. (93), we would have obtained a gaussian decay

$$\langle F(0)F(t) \rangle = \frac{\rho k^2}{4\pi} \cos(vt) e^{-\frac{t^2}{2\beta}}, \quad (113)$$

instead of an exponential decay with large  $\gamma$ -rate. This shows that collective effects are important even far from the critical point since they modify the large time behavior of the temporal correlation function. For the gravitational potential, Eq. (110) becomes

$$\gamma_k = \sqrt{\frac{2}{\beta m}} k F^{-1}\left(\frac{k_J^2}{k^2}\right). \quad (114)$$

Using  $F^{-1}(x) = \frac{1}{\sqrt{\pi}}(1-x) + \dots$  for  $x \rightarrow 1^-$  and  $F^{-1}(x) \sim \sqrt{-\ln x}$  for  $x \rightarrow 0$ , we find that

$$\gamma_k \simeq \sqrt{\frac{2}{\pi\beta m}} k_J \left(1 - \frac{k_J^2}{k^2}\right), \quad (k \rightarrow k_J^+) \quad (115)$$

$$\gamma_k \sim \frac{2k}{\sqrt{\beta m}} \ln\left(\frac{k}{k_J}\right)^{1/2}, \quad (k \rightarrow +\infty). \quad (116)$$

## J. Evolution of the spatial correlation function in the linear regime

To conclude this section on the kinetic theory of Hamiltonian systems with long-range interactions, we would like to discuss the time evolution of the spatial correlation function. Returning to the second equation of the BBGKY hierarchy (5) for the two-body correlation function and considering the case of a homogeneous medium, we obtain

$$\begin{aligned} & \frac{\partial P'_2}{\partial t} + \mathbf{v}_1 \frac{\partial P'_2}{\partial \mathbf{r}_1} + \mathbf{F}(2 \rightarrow 1) P_1(\mathbf{v}_2) \frac{\partial P_1}{\partial \mathbf{v}_1}(\mathbf{v}_1) \\ & + N \frac{\partial}{\partial \mathbf{v}_1} \int \mathbf{F}(3 \rightarrow 1) P'_2(\mathbf{x}_2, \mathbf{x}_3, t) P_1(\mathbf{v}_1) d^D \mathbf{x}_3 + (1 \leftrightarrow 2) = 0, \end{aligned} \quad (117)$$

while the first equation (4) gives  $\partial P_1 / \partial t = 0$  for  $N \rightarrow +\infty$ . We assume that, initially, no correlation is present among the particles. Then, for sufficiently small times (linear regime), the correlations will be small and we can neglect the integrals in Eq. (117). This yields

$$\frac{\partial P'_2}{\partial t} + \mathbf{v}_1 \frac{\partial P'_2}{\partial \mathbf{r}_1} + \mathbf{F}(2 \rightarrow 1) P_1(\mathbf{v}_2) \frac{\partial P_1}{\partial \mathbf{v}_1}(\mathbf{v}_1) + (1 \leftrightarrow 2) = 0. \quad (118)$$

We now assume that, initially,  $P_1 \sim \exp[-\beta m v^2/2]$  is the Maxwellian distribution. As we have seen previously, this distribution is conserved to leading order at later times. Introducing the correlation function  $h$  through the defining relation

$$P_2(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, t) = P_1(\mathbf{v}_1) P_1(\mathbf{v}_2) [1 + h(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{v}_1 - \mathbf{v}_2, t)], \quad (119)$$

we find that it satisfies the differential equation

$$\frac{\partial h}{\partial t} + \mathbf{u} \cdot \frac{\partial h}{\partial \mathbf{x}} = \beta m \mathbf{F}(2 \rightarrow 1) \cdot \mathbf{u}, \quad (120)$$

where  $\mathbf{x} = \mathbf{r}_1 - \mathbf{r}_2$  and  $\mathbf{u} = \mathbf{v}_1 - \mathbf{v}_2$ . For a stationary solution, we recover Eq. (I-53) of Paper I. Note also that Eq. (I-51) can be obtained from the static ( $\partial/\partial t = 0$ ) expression of Eq. (117) by keeping the integrals and assuming that  $P_1$  is Maxwellian.

Returning to Eq. (120), taking its Fourier transform and solving the resulting first order differential equation, we get

$$\hat{h}(\mathbf{k}, \mathbf{u}, t) = -\beta m^2 \hat{u}(k) (1 - e^{-i\mathbf{k} \cdot \mathbf{u} t}), \quad (121)$$

where we have assumed that, initially, the system is uncorrelated. Taking the inverse Fourier transform of Eq. (121), we finally obtain

$$h(\mathbf{x}, \mathbf{u}, t) = \beta m^2 [u(\mathbf{x} - \mathbf{u} t) - u(\mathbf{x})]. \quad (122)$$

It may be useful to discuss the HMF model explicitly. In that case, Eq. (120) becomes

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial \phi} = -\frac{k\beta}{2\pi} u \sin \phi. \quad (123)$$

The stationary solution of Eq. (123) is

$$h(\phi) = \frac{k\beta}{2\pi} \cos \phi. \quad (124)$$

This is similar to Eq. (I-66) of Paper I, but it does not display the expected divergence as we approach the critical point  $T_c$ . This is due to the neglect of the integrals in Eq. (117). Furthermore, it is noteworthy that the solution of Eq. (123) does not converge towards the stationary distribution (124) for  $t \rightarrow +\infty$ . Indeed, the explicit time-dependent solution of Eq. (123) is

$$h(\phi, u, t) = \frac{k\beta}{2\pi} \left[ \cos \phi - \cos(\phi - ut) \right], \quad (125)$$

which shows an oscillatory behavior. This is not really surprising since Eq. (125) is only valid for short times. In this linear regime

$$h(\phi, u, t) = -\frac{k\beta}{2\pi} \sin(\phi) ut \quad (t \rightarrow 0). \quad (126)$$

### III. KINETIC THEORY OF BROWNIAN SYSTEMS

#### A. The non-local Kramers equation

We shall now derive the kinetic equations describing a system of Brownian particles with long-range interactions defined by the stochastic equations (I-39)-(I-40) of Paper I. We shall use and generalize the method of Martzel & Aslangul [39, 40]. We start from the general Markov process

$$\begin{aligned} P_N(\mathbf{r}_1, \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N, t + \Delta t) &= \int d^D(\Delta \mathbf{r}_1) d^D(\Delta \mathbf{v}_1) \dots d^D(\Delta \mathbf{r}_N) d^D(\Delta \mathbf{v}_N) \\ &\times P_N(\mathbf{r}_1 - \Delta \mathbf{r}_1, \mathbf{v}_1 - \Delta \mathbf{v}_1, \dots, \mathbf{r}_N - \Delta \mathbf{r}_N, \mathbf{v}_N - \Delta \mathbf{v}_N, t) \\ &\times w(\mathbf{r}_1 - \Delta \mathbf{r}_1, \mathbf{v}_1 - \Delta \mathbf{v}_1, \dots, \mathbf{r}_N - \Delta \mathbf{r}_N, \mathbf{v}_N - \Delta \mathbf{v}_N | \Delta \mathbf{r}_1, \Delta \mathbf{v}_1, \dots, \Delta \mathbf{r}_N, \Delta \mathbf{v}_N). \end{aligned} \quad (127)$$

where  $w$  denotes the transition probability from one state to the other specified by the term in parenthesis. Using Eq. (I-39), it can be rewritten

$$\begin{aligned} &w(\mathbf{r}_1, \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N | \Delta \mathbf{r}_1, \Delta \mathbf{v}_1, \dots, \Delta \mathbf{r}_N, \Delta \mathbf{v}_N) = \\ &\delta(\Delta \mathbf{r}_1 - \mathbf{v}_1 \Delta t) \dots \delta(\Delta \mathbf{r}_N - \mathbf{v}_N \Delta t) \psi(\mathbf{r}_1, \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N | \Delta \mathbf{v}_1, \dots, \Delta \mathbf{v}_N). \end{aligned} \quad (128)$$

Then, the integration over  $\Delta \mathbf{r}_1 \dots \Delta \mathbf{r}_N$  is straightforward and yields

$$\begin{aligned} P_N(\mathbf{r}_1, \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N, t + \Delta t) &= \int d^D(\Delta \mathbf{v}_1) \dots d^D(\Delta \mathbf{v}_N) \\ &\times P_N(\mathbf{r}_1 - \mathbf{v}_1 \Delta t, \mathbf{v}_1 - \Delta \mathbf{v}_1, \dots, \mathbf{r}_N - \mathbf{v}_N \Delta t, \mathbf{v}_N - \Delta \mathbf{v}_N, t) \\ &\times \psi(\mathbf{r}_1 - \mathbf{v}_1 \Delta t, \mathbf{v}_1 - \Delta \mathbf{v}_1, \dots, \mathbf{r}_N - \mathbf{v}_N \Delta t, \mathbf{v}_N - \Delta \mathbf{v}_N | \Delta \mathbf{v}_1, \dots, \Delta \mathbf{v}_N), \end{aligned} \quad (129)$$

or equivalently

$$\begin{aligned}
P_N(\mathbf{r}_1 + \mathbf{v}_1 \Delta t, \mathbf{v}_1, \dots, \mathbf{r}_N + \mathbf{v}_N \Delta t, \mathbf{v}_N, t + \Delta t) &= \int d^D(\Delta \mathbf{v}_1) \dots d^D(\Delta \mathbf{v}_N) \\
&\times P_N(\mathbf{r}_1, \mathbf{v}_1 - \Delta \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N - \Delta \mathbf{v}_N, t) \\
&\times \psi(\mathbf{r}_1, \mathbf{v}_1 - \Delta \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N - \Delta \mathbf{v}_N | \Delta \mathbf{v}_1, \dots, \Delta \mathbf{v}_N).
\end{aligned} \tag{130}$$

Expanding the right hand side in Taylor series and introducing the Kramers-Moyal moments

$$\begin{aligned}
M_{n_1 \dots n_N}(\mathbf{r}_1, \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N) &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t n_1! \dots n_N!} \int d^D(\Delta \mathbf{v}_1) \dots d^D(\Delta \mathbf{v}_N) \\
&\times (-\Delta \mathbf{v}_1)^{n_1} \dots (-\Delta \mathbf{v}_N)^{n_N} \psi(\mathbf{r}_1, \mathbf{v}_1, \dots, \mathbf{r}_N, \mathbf{v}_N | \Delta \mathbf{v}_1, \dots, \Delta \mathbf{v}_N),
\end{aligned} \tag{131}$$

we get

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^N \mathbf{v}_i \frac{\partial P_N}{\partial \mathbf{r}_i} = \sum_{n_1 \dots n_N} \frac{\partial^{n_1}}{\partial \mathbf{v}_1^{n_1}} \dots \frac{\partial^{n_N}}{\partial \mathbf{v}_N^{n_N}} \{M_{n_1 \dots n_N} P_N\}, \tag{132}$$

where the sum runs over all indices such that  $\sum_i n_i \geq 1$ . For the stochastic process (I-39)-(I-40), only a few moments do not vanish, namely

$$M_{0 \dots n_i=1 \dots 0} = \xi \mathbf{v}_i + m \nabla_i U(\mathbf{r}_1, \dots, \mathbf{r}_N), \tag{133}$$

$$M_{0 \dots n_i=2 \dots 0} = D. \tag{134}$$

Substituting these results in Eq. (132), we obtain the  $N$ -body Fokker-Planck equation

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^N \left( \mathbf{v}_i \frac{\partial P_N}{\partial \mathbf{r}_i} + \mathbf{F}_i \frac{\partial P_N}{\partial \mathbf{v}_i} \right) = \sum_{i=1}^N \frac{\partial}{\partial \mathbf{v}_i} \left[ D \frac{\partial P_N}{\partial \mathbf{v}_i} + \xi P_N \mathbf{v}_i \right], \tag{135}$$

where  $\mathbf{F}_i = -m \nabla_i U(\mathbf{r}_1, \dots, \mathbf{r}_N)$  is the force by unit of mass acting on the  $i$ -th particle. The  $N$ -body Fokker-Planck equation decreases the free energy

$$F[P_N] = \langle E \rangle [P_N] - TS[P_N] \tag{136}$$

constructed with the average energy (I-8) and the entropy (I-26) defined in Paper I. Indeed, an explicit calculation yields

$$\dot{F} = - \sum_{i=1}^N \int \frac{1}{\xi P_N} \left( D \frac{\partial P_N}{\partial \mathbf{v}_i} + \xi P_N \mathbf{v}_i \right)^2 d^D \mathbf{r}_1 d^D \mathbf{v}_1 \dots d^D \mathbf{r}_N d^D \mathbf{v}_N \leq 0. \tag{137}$$

Since  $\dot{F} = 0$  at equilibrium, the term in bracket in Eq. (135) vanishes by virtue of Eq. (137). Since  $\partial/\partial t = 0$ , the advective term must also vanish, independently. From these two requirements, we find that the stationary solution of the  $N$ -body Fokker-Planck equation corresponds to the canonical distribution (I-42) which minimizes the free energy.

We can now obtain the equivalent of the BBGKY hierarchy for the reduced distribution functions. It reads

$$\begin{aligned}
\frac{\partial P_j}{\partial t} + \sum_{i=1}^j \mathbf{v}_i \frac{\partial P_j}{\partial \mathbf{r}_i} + \sum_{i=1}^j \sum_{k=1, k \neq i}^j \mathbf{F}(k \rightarrow i) \frac{\partial P_j}{\partial \mathbf{v}_i} + (N-j) \sum_{i=1}^j \int d^D \mathbf{x}_{j+1} \mathbf{F}(j+1 \rightarrow i) \frac{\partial P_{j+1}}{\partial \mathbf{v}_i} \\
= \sum_{i=1}^j \frac{\partial}{\partial \mathbf{v}_i} \left[ D \frac{\partial P_j}{\partial \mathbf{v}_i} + \xi P_j \mathbf{v}_i \right]
\end{aligned} \tag{138}$$

In particular, the first equation of the hierarchy is

$$\frac{\partial P_1}{\partial t} + \mathbf{v}_1 \frac{\partial P_1}{\partial \mathbf{r}_1} + (N-1) \int d^D \mathbf{x}_2 \mathbf{F}(2 \rightarrow 1) \frac{\partial P_2}{\partial \mathbf{v}_1} = \frac{\partial}{\partial \mathbf{v}_1} \left[ D \frac{\partial P_1}{\partial \mathbf{v}_1} + \xi P_1 \mathbf{v}_1 \right]. \quad (139)$$

We can similarly obtain an equation for the two-body correlation function. Implementing the decomposition (I-14)-(I-15) of Paper I, neglecting the cumulant of the three-body correlation function, and taking the thermodynamical limit defined in Paper I, we find that  $P_2' \sim 1/N$ . Therefore, in the limit  $N \rightarrow +\infty$ , we can make the mean-field approximation

$$P_2(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, t) = P_1(\mathbf{r}_1, \mathbf{v}_1, t) P_1(\mathbf{r}_2, \mathbf{v}_2, t). \quad (140)$$

We thus obtain

$$\frac{\partial P_1}{\partial t} + \mathbf{v}_1 \frac{\partial P_1}{\partial \mathbf{r}_1} + \langle \mathbf{F} \rangle_1 \frac{\partial P_1}{\partial \mathbf{v}_1} = \frac{\partial}{\partial \mathbf{v}_1} \left[ D \frac{\partial P_1}{\partial \mathbf{v}_1} + \xi P_1 \mathbf{v}_1 \right], \quad (141)$$

where

$$\langle \mathbf{F} \rangle_1 = -Nm \int d^D \mathbf{r}_2 d^D \mathbf{v}_2 \frac{\partial u}{\partial \mathbf{r}_1}(\mathbf{r}_1 - \mathbf{r}_2) P_1(\mathbf{r}_2, \mathbf{v}_2, t). \quad (142)$$

Introducing the distribution function  $f = NmP_1$ , this can be rewritten

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \langle \mathbf{F} \rangle \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \left[ D \left( \frac{\partial f}{\partial \mathbf{v}} + \beta m f \mathbf{v} \right) \right], \quad (143)$$

where

$$\langle \mathbf{F} \rangle = -\nabla \Phi = - \int d^D \mathbf{r}' \rho(\mathbf{r}', t) \frac{\partial u}{\partial \mathbf{r}}(\mathbf{r} - \mathbf{r}') \quad (144)$$

is the mean-field force acting on a particle and we have set  $\xi = D\beta m$ . Equation (143) is a non-local Kramers equation. It decreases the free energy

$$F[f] = E[f] - TS[f] = \frac{1}{2} \int f v^2 d^D \mathbf{r} d^D \mathbf{v} + \frac{1}{2} \int \rho \Phi d^D \mathbf{r} + T \int f \ln f d^D \mathbf{r} d^D \mathbf{v}, \quad (145)$$

which plays the role of a Lyapunov functional [18]. The free energy (145) is the resulting expression of Eq. (136) in the mean-field approximation. We note that the non-local Kramers equation (143) is obtained at the same level of approximation (i.e. for  $N \rightarrow +\infty$ ) as the Vlasov equation (7) for Hamiltonian systems. The introduction of a friction and a random force in the equations of motion (I-39)-(I-40) yields a “collision term” of the Fokker-Planck form in the right hand side of Eq. (143). This “collision” term selects the mean-field Maxwell-Boltzmann equilibrium distribution (I-24) among the infinite class of stationary solutions of the Vlasov equation (left hand side). This mean-field Maxwell-Boltzmann distribution extremizes the free energy (145) at fixed mass. Furthermore, only minima of free energy are linearly dynamically stable via Eq. (143) [18]. We should also contrast the non-local Fokker-Planck equation (143) to the local Fokker-Planck Eq. (62). Although they look similar, their physical content is quite different. Indeed, Eq. (55) describes the motion of a *single* test particle in a thermal bath at statistical equilibrium while Eq. (143) describes the evolution of the *whole* system of  $N$  Brownian particles in interaction out of equilibrium. Therefore, Eq. (55) is a local differential equation while Eq. (143) is non-local due to the mean-field force produced by the distribution function  $f(\mathbf{r}, \mathbf{v}, t)$  evolving in time. In addition, in the present case, the diffusion coefficient  $D$  is *given* in Eq. (I-40) while in the test particle approach it is derived from the Hamiltonian dynamics at the order  $1/N$  for  $N \gg 1$ .

## B. The non-local Smoluchowski equation

In the strong friction limit  $\xi \rightarrow +\infty$ , or equivalently for large times  $t \gg \xi^{-1}$ , it is possible to neglect the inertial term in Eq. (I-40). In that case, we are led to consider a system of  $N$  Brownian particles in interaction described by the coupled stochastic equations in physical space

$$\frac{d\mathbf{r}_i}{dt} = -\mu m^2 \nabla_i U(\mathbf{r}_1, \dots, \mathbf{r}_N) + \sqrt{2D_*} \mathbf{R}_i(t), \quad (146)$$

where  $\mu = 1/m\xi$  is the mobility and  $D_* = D/\xi^2 = T/m\xi$  is the diffusion coefficient in physical space. The Einstein relation reads  $\mu = \beta D_*$ . We can now repeat the above procedure to obtain the kinetic equations governing the evolution of the Brownian system in the overdamped regime. For a general Markovian process in physical space, we have

$$P_N(\mathbf{r}_1, \dots, \mathbf{r}_N, t + \Delta t) = \int d^D(\Delta \mathbf{r}_1) \dots d^D(\Delta \mathbf{r}_N) P_N(\mathbf{r}_1 - \Delta \mathbf{r}_1, \dots, \mathbf{r}_N - \Delta \mathbf{r}_N, t) \\ \times w(\mathbf{r}_1 - \Delta \mathbf{r}_1, \dots, \mathbf{r}_N - \Delta \mathbf{r}_N | \Delta \mathbf{r}_1, \dots, \Delta \mathbf{r}_N) \quad (147)$$

where  $w$  denotes the transition probability. Expanding the right hand side in Taylor series and introducing the Kramers-Moyal moments

$$M_{n_1 \dots n_N}(\mathbf{r}_1, \dots, \mathbf{r}_N) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t n_1! \dots n_N!} \int d^D(\Delta \mathbf{r}_1) \dots d^D(\Delta \mathbf{r}_N) (-\Delta \mathbf{r}_1)^{n_1} \dots (-\Delta \mathbf{r}_N)^{n_N} \\ \times w(\mathbf{r}_1, \dots, \mathbf{r}_N | \Delta \mathbf{r}_1, \dots, \Delta \mathbf{r}_N) \quad (148)$$

we get

$$\frac{\partial P_N}{\partial t} = \sum_{n_1 \dots n_N} \frac{\partial^{n_1}}{\partial \mathbf{r}_1^{n_1}} \dots \frac{\partial^{n_N}}{\partial \mathbf{r}_N^{n_N}} \{M_{n_1 \dots n_N} P_N\} \quad (149)$$

where the sum runs over all indices such that  $\sum_i n_i \geq 1$ . For the stochastic process (146), only a few moments do not vanish, namely

$$M_{0 \dots n_i=1 \dots 0} = \mu m^2 \nabla_i U(\mathbf{r}_1, \dots, \mathbf{r}_N), \quad (150)$$

$$M_{0 \dots n_i=2 \dots 0} = D_*. \quad (151)$$

Substituting these results in Eq. (149), we obtain the  $N$ -body Fokker-Planck equation

$$\frac{\partial P_N}{\partial t} = \sum_{i=1}^N \frac{\partial}{\partial \mathbf{r}_i} \left[ D_* \frac{\partial P_N}{\partial \mathbf{r}_i} + \mu m^2 P_N \frac{\partial}{\partial \mathbf{r}_i} U(\mathbf{r}_1, \dots, \mathbf{r}_N) \right]. \quad (152)$$

The stationary solutions of this equation correspond to the configurational part of the canonical distribution (I-44). Again, it is possible to derive the equivalent of the BBGKY hierarchy for the reduced distribution functions. It reads

$$\frac{\partial P_j}{\partial t} = \sum_{i=1}^j \frac{\partial}{\partial \mathbf{r}_i} \left[ D_* \frac{\partial P_j}{\partial \mathbf{r}_i} + \mu m^2 \sum_{k=1, k \neq i}^j P_j \frac{\partial u_{ik}}{\partial \mathbf{r}_i} + \mu m^2 (N-j) \int P_{j+1} \frac{\partial u_{i,j+1}}{\partial \mathbf{r}_i} d^D \mathbf{r}_{j+1} \right]. \quad (153)$$

Introducing the decomposition (I-14)-(I-15), neglecting the three-body correlation function and considering only terms of order  $1/N$  or larger in the thermodynamic limit  $N \rightarrow +\infty$ , we obtain

$$\frac{\partial P_1}{\partial t} = \frac{\partial}{\partial \mathbf{r}_1} \left[ D_* \frac{\partial P_1}{\partial \mathbf{r}_1} + \mu N m^2 P_1(\mathbf{r}_1, t) \int P_1(\mathbf{r}_2, t) \frac{\partial u_{12}}{\partial \mathbf{r}_1} d^D \mathbf{r}_2 + \mu N m^2 \int P_2'(\mathbf{r}_1, \mathbf{r}_2, t) \frac{\partial u_{12}}{\partial \mathbf{r}_1} d^D \mathbf{r}_2 \right], \quad (154)$$

$$\begin{aligned} \frac{\partial P_2'}{\partial t} = \frac{\partial}{\partial \mathbf{r}_1} \left[ D_* \frac{\partial P_2'}{\partial \mathbf{r}_1} + \mu m^2 P_1(\mathbf{r}_1, t) P_1(\mathbf{r}_2, t) \frac{\partial u_{12}}{\partial \mathbf{r}_1} + N \mu m^2 P_2'(\mathbf{r}_1, \mathbf{r}_2, t) \int P_1(\mathbf{r}_3, t) \frac{\partial u_{13}}{\partial \mathbf{r}_1} d^D \mathbf{r}_3 \right. \\ \left. + N \mu m^2 P_1(\mathbf{r}_1, t) \int P_2'(\mathbf{r}_2, \mathbf{r}_3, t) \frac{\partial u_{13}}{\partial \mathbf{r}_1} d^D \mathbf{r}_3 \right] + (1 \leftrightarrow 2) \end{aligned} \quad (155)$$

The stationary solutions of these equations coincide with the equations of the equilibrium BBGKY-like hierarchy (see Paper I) as expected. In the limit  $N \rightarrow \infty$ , we can make the mean-field approximation

$$P_2(\mathbf{r}_1, \mathbf{r}_2, t) = P_1(\mathbf{r}_1, t) P_1(\mathbf{r}_2, t). \quad (156)$$

The equation for the density then reduces to

$$\frac{\partial P_1}{\partial t} = \frac{\partial}{\partial \mathbf{r}_1} \left[ D_* \frac{\partial P_1}{\partial \mathbf{r}_1} + \mu m P_1 \frac{\partial \Phi}{\partial \mathbf{r}_1} \right], \quad (157)$$

which can be written

$$\frac{\partial \rho}{\partial t} = \nabla \cdot [D_* (\nabla \rho + \beta m \rho \nabla \Phi)], \quad (158)$$

where  $\Phi(\mathbf{r}, t)$  is related to  $\rho(\mathbf{r}, t)$  as in Eq. (144). This is the non-local Smoluchowski equation. This equation decreases the free energy

$$F[\rho] = \frac{1}{2} \int \rho \Phi d^D \mathbf{r} + T \int \rho \ln \rho d^D \mathbf{r}, \quad (159)$$

which plays the role of a Lyapunov functional. The Smoluchowski equation can also be derived directly from the Kramers equation in the strong friction limit  $\xi \rightarrow +\infty$  [41]. This can be done by working out the moments equations of Eq. (143) as in [18] or by using a Chapman-Enskog expansion as in [42]. To leading order in  $1/\xi$ , the velocity distribution is Maxwellian

$$f(\mathbf{r}, \mathbf{v}, t) = \left( \frac{\beta m}{2\pi} \right)^{D/2} \rho(\mathbf{r}, t) e^{-\beta m \frac{v^2}{2}} + O(\xi^{-1}), \quad (160)$$

and the evolution of  $\rho(\mathbf{r}, t)$  is given by Eq. (158). The free energy (159) can be obtained from Eq. (145) by using the approximate expression (160) of the distribution function to express the free energy as a functional of  $\rho$ .

Considering the linear dynamical stability of a spatially homogeneous distribution of particles with respect to the non-local Smoluchowski equation (158) with  $\Phi = \rho * u$ , we immediately get the dispersion relation

$$i\xi\omega = \frac{T}{m} k^2 + (2\pi)^D \hat{u}(k) k^2 \rho. \quad (161)$$

We note that the condition of instability (corresponding to  $i\omega < 0$ ) is the same as for an isothermal distribution described by the Euler or the Vlasov equation (see Secs. IIC and IID). However, the evolution of the perturbation is different. In the unstable regime  $\omega = i\lambda$  with  $\lambda > 0$ , the perturbation grows exponentially rapidly as  $\delta\rho \sim e^{\lambda t}$ . In the stable regime  $\omega = -i\gamma$  with  $\gamma > 0$ , the perturbation decreases exponentially rapidly as  $\delta\rho \sim e^{-\gamma t}$ .

### C. Evolution of the spatial correlations

In this section, we study the development of the spatial correlations (at the order  $1/N$ ) for a Brownian system in the overdamped regime. For a homogeneous system, Eq. (155) for the two-body correlation function reduces to

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial \mathbf{r}_1} \left[ D_* \frac{\partial h}{\partial \mathbf{r}_1} + \mu m^2 \frac{\partial u_{12}}{\partial \mathbf{r}_1} + \mu m \rho \int h(\mathbf{r}_2 - \mathbf{r}_3, t) \frac{\partial u_{13}}{\partial \mathbf{r}_1} d^D \mathbf{r}_3 \right] + (1 \leftrightarrow 2), \quad (162)$$

or, equivalently,

$$\frac{\partial h}{\partial t} = 2D_* \Delta \left[ h(\mathbf{x}, t) + \beta m^2 u(\mathbf{x}) + \beta m \rho \int h(\mathbf{y}, t) u(\mathbf{x} - \mathbf{y}) d^D \mathbf{y} \right], \quad (163)$$

where  $\mathbf{x} = \mathbf{r}_1 - \mathbf{r}_2$ . Taking its Fourier transform, we get

$$\frac{\partial \hat{h}}{\partial t} + 2D_* k^2 [1 + (2\pi)^D \beta m \rho \hat{u}(k)] \hat{h} = -2D_* \beta m^2 k^2 \hat{u}(k). \quad (164)$$

This equation is easily integrated in time to yield

$$\hat{h}(k, t) = \hat{h}_{eq}(k) \left\{ 1 - e^{-2D_* k^2 [1 + (2\pi)^D \beta m \rho \hat{u}(k)] t} \right\}, \quad (165)$$

where  $\hat{h}_{eq}(k)$  is the equilibrium value of the correlation function (in Fourier space) given by Eq. (I-54) of Paper I. For the BMF model (one mode), we find that

$$h(\phi, t) = \frac{\beta k / 2\pi}{1 - \beta / \beta_c} \left[ 1 - e^{-2D_* (1 - \beta / \beta_c) t} \right] \cos \phi. \quad (166)$$

For  $t \rightarrow +\infty$ , the correlation function relaxes towards its equilibrium form (I-66). Note however that the relaxation time diverges as we approach the critical point since  $t_{relax} \sim (1 - \beta / \beta_c)^{-1}$ .

## IV. GENERALIZED KINETIC EQUATIONS AND EFFECTIVE THERMODYNAMICS

### A. Generalized Kramers and Smoluchowski equations

We shall introduce a class of stochastic processes leading to generalized Fokker-Planck equations. These equations are associated with an effective thermodynamical formalism in  $\mu$ -space. Let us first consider the case of non-interacting Langevin particles described by the stochastic process

$$\frac{d\mathbf{r}}{dt} = -\mu \nabla \Phi_{ext}(\mathbf{r}) + \sqrt{K(\mathbf{r}, t)} \mathbf{R}(t), \quad (167)$$

where  $\mathbf{R}(t)$  is a white noise and  $\Phi_{ext}(\mathbf{r})$  an external potential. Since the function in front of  $\mathbf{R}(t)$  depends on the position, the last term in Eq. (167) can be interpreted as a multiplicative noise. The corresponding Fokker-Planck equation is

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \Delta (K(\mathbf{r}, t) \rho) + \mu \nabla (\rho \nabla \Phi_{ext}). \quad (168)$$

When  $K$  is constant, one recovers the usual Fokker-Planck equation with a diffusion coefficient  $D = K/2$ . Recently, there was some interest for the nonlinear Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \Delta(D\rho^q) + \mu \nabla(\rho \nabla \Phi_{ext}), \quad (169)$$

that arises in connection with Tsallis generalized thermodynamics, see e.g. [43]. Defining a generalized temperature  $T = 1/\beta$  through the Einstein-like relation  $T = D/\mu$ , we can rewrite the foregoing equation in the form

$$\frac{\partial \rho}{\partial t} = \nabla[D(\nabla \rho^q + \beta \rho \nabla \Phi_{ext})]. \quad (170)$$

It is easy to verify that the nonlinear Fokker-Planck equation decreases the Lyapunov functional

$$F[\rho] = \int \rho \Phi_{ext} d^D \mathbf{r} + \frac{T}{q-1} \int (\rho^q - \rho) d^D \mathbf{r}, \quad (171)$$

which can be interpreted as a free energy  $F = E - TS$  associated with the Tsallis entropy  $S_q = -\frac{1}{q-1} \int (\rho^q - \rho) d^D \mathbf{r}$ . Furthermore, the stationary solutions of this equation are given by a  $q$ -distribution

$$\rho = \left[ \alpha - \frac{\beta(q-1)}{q} \Phi_{ext} \right]^{\frac{1}{q-1}}, \quad (172)$$

which minimizes the Tsallis free energy at fixed mass. In an attempt to justify the nonlinear Fokker-Planck equation from a microscopic model, Borland [44] proposed to consider the generalized stochastic process

$$\frac{d\mathbf{r}}{dt} = -\mu \nabla \Phi_{ext}(\mathbf{r}) + \sqrt{2D} \rho(\mathbf{r}, t)^{(q-1)/2} \mathbf{R}(t). \quad (173)$$

The last term is a multiplicative noise which depends on  $\mathbf{r}$  and  $t$  through the density  $\rho(\mathbf{r}, t)$ . Therefore, there is a feedback from the macroscopic dynamics. For this stochastic process,  $K(\mathbf{r}, t) = 2D\rho^{q-1}$  and the Fokker-Planck equation (168) takes the form (169) where the diffusion coefficient depends on the density.

In a previous work [18], we remarked that the nonlinear Fokker-Planck equation (170) is a particular case of a larger class of generalized Fokker-Planck equations associated with generalized entropy functionals encompassing Tsallis entropy. Similar observations have been made independently by Frank [45] and Kaniadakis [46]. These equations can be written

$$\frac{\partial \rho}{\partial t} = \nabla \{ D[\rho C''(\rho) \nabla \rho + \beta \rho \nabla \Phi_{ext}] \}, \quad (174)$$

where  $C$  is a convex function, and they monotonically decrease the functional

$$F = \int \rho \Phi_{ext} d^D \mathbf{r} + T \int C(\rho) d^D \mathbf{r}, \quad (175)$$

which can be interpreted as a generalized free energy. When  $C(\rho) = \rho \ln \rho$ , we recover the usual Fokker-Planck equation associated with the Boltzmann free energy and when  $C(\rho) = (\rho^q - \rho)/(q-1)$  we recover the nonlinear Fokker-Planck equation associated with the Tsallis free energy. Comparing Eq. (174) with Eq. (168) we find that

$$\mu = D\beta, \quad \frac{1}{2} \nabla(K\rho) = D\rho C''(\rho) \nabla \rho. \quad (176)$$

The second equation is equivalent to

$$\frac{1}{2}K\rho = \int^\rho D\rho C''(\rho)d\rho. \quad (177)$$

Integrating by parts, we get

$$\frac{1}{2}K\rho = D[\rho C'(\rho) - C(\rho)], \quad (178)$$

so that, finally,

$$K(\mathbf{r}, t) = 2D\rho \left[ \frac{C(\rho)}{\rho} \right]'. \quad (179)$$

Therefore, a stochastic process leading to the generalized Fokker-Planck equation (174) is given by

$$\frac{d\mathbf{r}}{dt} = -\mu\nabla\Phi_{ext}(\mathbf{r}) + \sqrt{2D\rho \left[ \frac{C(\rho)}{\rho} \right]'} \mathbf{R}(t), \quad (180)$$

where  $\mathbf{R}(t)$  is a white noise. We note in particular that the strength of the stochastic force depends on the local density of particles. When  $C(\rho) = \rho \ln \rho$ , we recover the usual Langevin equation and when  $C(\rho) = (\rho^q - \rho)/(q - 1)$  we recover the stochastic process considered by Borland [44]. The same arguments can be developed in velocity space (taking into account the inertia of the particles), see [47], leading to the generalized Kramers equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \langle \mathbf{F} \rangle_{ext} \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \left[ D \left( f C''(f) \frac{\partial f}{\partial \mathbf{v}} + \beta f \mathbf{v} \right) \right]. \quad (181)$$

## B. Generalized non-local Kramers and Smoluchowski equations

We shall now consider the generalization of the previous approach to the case of particles in interaction. We thus consider the  $N$  coupled stochastic equations

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \quad (182)$$

$$\frac{d\mathbf{v}_i}{dt} = -\xi \mathbf{v}_i - \nabla_i U(\mathbf{r}_1, \dots, \mathbf{r}_N) + \sqrt{2Df_i \left[ \frac{C(f_i)}{f_i} \right]'} \mathbf{R}_i(t), \quad (183)$$

where  $f_i = f(\mathbf{r}_i, \mathbf{v}_i, t)$ . We shall say that these equations describe a gas of Langevin particles in interaction [48]. We reserve the term of Brownian particles for usual diffusion when  $C(f) = f \ln f$  is the Boltzmann function (see Sec. III A). Using exactly the same steps as in Sec. III A, we can derive the generalized  $N$ -body Fokker-Planck equation

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^N \left( \mathbf{v}_i \frac{\partial P_N}{\partial \mathbf{r}_i} + \mathbf{F}_i \frac{\partial P_N}{\partial \mathbf{v}_i} \right) = \sum_{i=1}^N \frac{\partial}{\partial \mathbf{v}_i} \left[ \frac{\partial}{\partial \mathbf{v}_i} \left( Df_i \left[ \frac{C(f_i)}{f_i} \right]' P_N \right) + \xi P_N \mathbf{v}_i \right]. \quad (184)$$

Implementing a mean-field approximation, we arrive at the generalized non-local Kramers equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \langle \mathbf{F} \rangle \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \left[ D \left( f C''(f) \frac{\partial f}{\partial \mathbf{v}} + \beta f \mathbf{v} \right) \right], \quad (185)$$

where  $\langle \mathbf{F} \rangle$  is the mean-field force (144) and we have set  $\xi = D\beta$ . This equation decreases the Lyapunov functional

$$F[f] = \frac{1}{2} \int f v^2 d^D \mathbf{r} d^D \mathbf{v} + \frac{1}{2} \int \rho \Phi d^D \mathbf{r} + T \int C(f) d^D \mathbf{r} d^D \mathbf{v}, \quad (186)$$

which can be interpreted as a generalized free energy.

Considering the strong friction limit  $\xi \rightarrow +\infty$ , or the limit of large times  $t \gg \xi^{-1}$ , we can derive a generalized Smoluchowski equation. This is done in [18] from the moment equations of Eq. (185) and in [42] by using a Chapman-Enskog expansion. The generalized Smoluchowski equation can be written

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} (\nabla p + \rho \nabla \Phi) \right]. \quad (187)$$

The fluid is barotropic in the sense that  $p = p(\rho)$  where the equation of state is entirely specified by the function  $C(f)$ . For the Boltzmann entropy, we recover the ordinary Smoluchowski equation (158) with  $p = \rho \frac{T}{m}$ . The generalized Smoluchowski equation (187) decreases the Lyapunov functional

$$F[\rho] = \int \rho \int_0^\rho \frac{p(\rho')}{\rho'^2} d\rho' d^D \mathbf{r} + \frac{1}{2} \int \rho \Phi d^D \mathbf{r}, \quad (188)$$

which can be interpreted as a generalized free energy. It can be derived from the generalized free energy (186) by using the leading order expression of the velocity distribution  $C'(f) = -\beta[\frac{v^2}{2} + \lambda(\mathbf{r}, t)] + O(1/\xi)$  to express  $F[f]$  as a functional of  $\rho$  (see [42] for details). Considering the dynamical stability of a spatially homogeneous solution of the generalized Smoluchowski equation (187) with  $\Phi = \rho * u$ , we get the dispersion relation

$$i\omega\xi = c_s^2 k^2 + (2\pi)^D \hat{u}(k) k^2 \rho, \quad (189)$$

with  $c_s^2 = p'(\rho)$  which generalizes (161). For the damped Euler equations [18], the term  $i\omega\xi$  is replaced by  $\omega(\omega + i\xi)$ . For  $\xi = 0$  we recover (28) and for  $\xi \rightarrow +\infty$  we recover (189).

We can also obtain a form of generalized Smoluchowski equation by starting directly from the coupled stochastic equations in physical space

$$\frac{d\mathbf{r}_i}{dt} = -\mu \nabla_i U(\mathbf{r}_1, \dots, \mathbf{r}_N) + \sqrt{2D_* \rho_i \left[ \frac{C(\rho_i)}{\rho_i} \right]'} \mathbf{R}_i(t), \quad (190)$$

where  $\rho_i \equiv \rho(\mathbf{r}_i, t)$  and  $\mathbf{R}_i(t)$  is a white noise acting independently on each particle. We note that these equations cannot be obtained from Eqs. (182)-(183) by simply neglecting the inertial term, as is done in the standard Brownian case. Applying steps similar to those of Sec. IIIB, we obtain the generalized  $N$ -body Fokker-Planck equation

$$\frac{\partial P_N}{\partial t} = \sum_{i=1}^N \frac{\partial}{\partial \mathbf{r}_i} \left[ \frac{\partial}{\partial \mathbf{r}_i} \left( D_* \rho_i \left[ \frac{C(\rho_i)}{\rho_i} \right]' P_N \right) + \mu P_N \frac{\partial}{\partial \mathbf{r}_i} U(\mathbf{r}_1, \dots, \mathbf{r}_N) \right]. \quad (191)$$

Implementing a mean-field approximation, we finally get the generalized non-local Smoluchowski equation

$$\frac{\partial \rho}{\partial t} = \nabla [D_*(\rho C''(\rho) \nabla \rho + \beta \rho \nabla \Phi)], \quad (192)$$

where  $\Phi$  is related to  $\rho$  as in Eq. (144) and we have set  $\mu = D_*\beta$ . This equation decreases the Lyapunov functional

$$F[\rho] = \frac{1}{2} \int \rho \Phi d^D \mathbf{r} + T \int C(\rho) d^D \mathbf{r}, \quad (193)$$

which can be interpreted as an effective free energy.

We note that the stationary solution of the generalized Fokker-Planck equation (191) is determined by the integro-differential equation

$$\frac{\partial}{\partial \mathbf{r}_i} \left( D_* \rho_i \left[ \frac{C(\rho_i)}{\rho_i} \right]' P_N \right) + \mu P_N \frac{\partial}{\partial \mathbf{r}_i} U(\mathbf{r}_1, \dots, \mathbf{r}_N) = \mathbf{0}, \quad (194)$$

where  $\rho(\mathbf{r}_i) = \int P_N d^D \mathbf{r}_1 \dots d^D \mathbf{r}_{i-1} d^D \mathbf{r}_{i+1} \dots d^D \mathbf{r}_N$ . Contrary to the case of Brownian particles studied in Sec. IIIB, the equilibrium  $N$ -body distribution does not seem to have a simple form. In particular, it does not seem to be possible to obtain a simple generalization of the canonical distribution in  $\Gamma$ -space by this approach. However, if we implement a mean-field approximation, a notion of generalized thermodynamics emerges in  $\mu$ -space, as we have seen. In this context, generalized free energies are Lyapunov functionals associated with generalized Fokker-Planck equations. The fact that we cannot obtain a simple form of equilibrium distribution in  $\Gamma$ -space may suggest that the generalized thermodynamical formalism developed in  $\mu$ -space is just *effective*.

### C. Generalized Landau equation

In [49], we have proposed to consider a generalized class of Landau equations of the form

$$\frac{\partial f}{\partial t} = \pi(2\pi)^D m \frac{\partial}{\partial v^\mu} \int d^D \mathbf{v}_1 d^D \mathbf{k} k^\mu k^\nu \hat{u}(\mathbf{k})^2 \delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}_1)] f f_1 \left( C''(f) \frac{\partial f}{\partial v^\nu} - C''(f_1) \frac{\partial f_1}{\partial v_1^\nu} \right) \quad (195)$$

where  $C$  is a convex function. For  $C = f \ln f$ , we recover the ordinary Landau equation (39) as a particular case. As shown in [49], Eq. (195) can be derived from the generalized Boltzmann equation introduced by Kaniadakis [46] in a weak deflexion limit. This equation conserves mass and energy and increases the generalized entropy  $S = - \int C(f) d^D \mathbf{r} d^D \mathbf{v}$ . This corresponds to a microcanonical structure while the Fokker-Planck equation has a canonical structure as it decreases the free energy at fixed mass and temperature.

In [49], we have also considered a test particle approach and showed the connection between the generalized Landau equation and the generalized Fokker-Planck equation. Explicit expressions of the diffusion coefficient have been obtained in  $D = 3$  for different entropies  $C(f)$ . We consider here the case  $D = 1$  and show that the generalized Fokker-Planck equation takes a relatively simple form. First, the generalized Landau equation in  $D = 1$  reads

$$\frac{\partial f}{\partial t} = K \frac{\partial}{\partial v} \int dv_1 \delta(v - v_1) f f_1 \left( C''(f) \frac{\partial f}{\partial v} - C''(f_1) \frac{\partial f_1}{\partial v_1} \right), \quad (196)$$

where  $K$  is given by Eq. (47). Once again, we see that the collision term vanishes in  $D = 1$ . In the test particle approach, we must consider that  $f_1$  is fixed to the distribution of the bath, i.e.  $f_1 = f_0(v)$ . This yields

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left\{ D(v) \left[ f C''(f) \frac{\partial f}{\partial v} - f \frac{dC'(f_0)}{dv} \right] \right\}, \quad (197)$$

where  $D(v) = K f_0(v)$ . This equation governs the evolution of the density probability  $f(\mathbf{v}, t)$  of a single test particle in a bath of field particles with fixed distribution  $f_0(v)$ . We note that, for  $t \rightarrow +\infty$ , the test particle relaxes to the distribution of the bath, i.e.  $f = f_0$ .

## V. CONCLUSION

In this paper (Paper II), we have discussed the kinetic theory of Hamiltonian and Brownian systems with long-range interactions. The statistical equilibrium states of these systems have been considered in Paper I. For Hamiltonian systems, in the  $N \rightarrow +\infty$  limit, we get the Vlasov equation. This equation admits an infinity of stationary solutions. One of them will be attained (on a coarse-grained scale) as a result of a violent relaxation on a short time scale of the order of the dynamical time  $\sim t_D$ . This metaequilibrium state, or quasi-stationary state (QSS), which depends on the initial conditions (due to the Casimir invariants) and on the efficiency of the mixing process (ergodicity), is usually difficult to predict [28]. The statistical equilibrium state, due to the development of correlations for finite  $N$  systems, is reached on a much longer timescale  $t_{relax} \sim N^\delta t_D$  increasing with the number of particles. For two and three-dimensional homogeneous systems, we can develop the kinetic theory at order  $1/N$  and get the Lenard-Balescu equation. This equation converges towards the Maxwellian distribution establishing  $\delta = 1$ . For inhomogeneous gravitational systems, we get the Vlasov-Landau-Poisson equation when the collisions are treated as local and collective effects are neglected. This equation converges towards the mean-field Maxwell-Boltzmann distribution. Its dynamical stability (with respect to the Landau equation) coincides with the thermodynamical stability criterion in the microcanonical ensemble (entropy maximum). Because of logarithmic divergences of the diffusion coefficient, the relaxation time scales as  $t_{relax} \sim (N/\ln N)t_D$ . Finally, for one-dimensional systems (and 2D point vortices with monotonic angular velocity profile), the collision term vanishes at the order  $1/N$  so that the evolution is due to higher order correlations. This implies  $\delta > 1$  as observed by [12] for the HMF model. For Brownian systems, in the  $N \rightarrow +\infty$  limit, we get the non-local Kramers and Smoluchowski equations. The mean-field Maxwell-Boltzmann distribution is the only stationary solution of these equations. Its dynamical stability (with respect to the Fokker-Planck equation) coincides with the thermodynamical stability criterion in the canonical ensemble (minimum of free energy).

We have also *formally* introduced generalized kinetic equations and showed that they were associated with a generalized thermodynamical framework in  $\mu$  space. We have introduced these equations (185), (187) and (192) from a specific class of stochastic processes (182)-(183) and (190) but we believe that these equations can have interest in much more general situations. They can be viewed as effective kinetic equations attempting to describe complex media. They may be useful when we are not in the strict conditions of applicability of standard kinetic theories. For example, there are situations in which the two-body distribution function cannot be factorized as a product of two one-body distribution functions

plus a small correction. In that case, the system is not described by the “ordinary” kinetic equations presented in Secs. II and III. On the other hand, the dynamics of complex systems can be altered by microscopic constraints (hidden constraints) that modify the form of transition probabilities. Generalized kinetic equation can then be of interest to describe (at least heuristically) these non-ideal situations. One property of these generalized kinetic equations is to exhibit anomalous diffusion (since the diffusion coefficient depends on the density) and indeed, anomalous diffusion is observed in complex media. In that case, this is associated with a complex geometrical structure of phase space (e.g., multifractal in the case of the Tsallis entropy). Note that anomalous diffusion can also be due to the rapid decay of the diffusion coefficient with the velocity as in the Fokker-Planck equation (62). This is another, completely independent, reason for anomalous diffusion [11]. In that case, the Fokker-Planck equation is derived from the pure Hamiltonian dynamics and there is no relation with generalized thermodynamics. These two approaches apply to different regimes or different systems.

## APPENDIX A: THE LANDAU EQUATION AND THE TEMPORAL CORRELATION FUNCTION OF THE FORCE WITHOUT COLLECTIVE EFFECTS

In this Appendix, we simplify the collision term appearing in the kinetic equation (38). For a homogeneous system, we need to compute the memory function

$$M^{\mu\nu} = \int_0^{+\infty} dt \int d^D \mathbf{r}_1 F^\mu(1 \rightarrow 0, 0) F^\nu(1 \rightarrow 0, t). \quad (\text{A1})$$

The force created by particle 1 on particle 0 at time  $t$  can be written

$$\mathbf{F}(1 \rightarrow 0, t) = -im \int \mathbf{k} \hat{u}(\mathbf{k}) e^{i\mathbf{k}(\mathbf{r}(t) - \mathbf{r}_1(t))} d^D \mathbf{k}. \quad (\text{A2})$$

Making a linear trajectory approximation  $\mathbf{r}(t) = \mathbf{r} + \mathbf{v}t$  and  $\mathbf{r}_1(t) = \mathbf{r}_1 + \mathbf{v}_1 t$ , we get

$$\mathbf{F}(1 \rightarrow 0, t) = -im \int \mathbf{k} \hat{u}(\mathbf{k}) e^{i\mathbf{k}(\mathbf{x} + \mathbf{u}t)} d^D \mathbf{k}, \quad (\text{A3})$$

where  $\mathbf{x} = \mathbf{r} - \mathbf{r}_1$  and  $\mathbf{u} = \mathbf{v} - \mathbf{v}_1$ . Now, the memory function is easily calculated and yields

$$M^{\mu\nu} = \pi(2\pi)^D m^2 \int k^\mu k^\nu \delta(\mathbf{k} \cdot \mathbf{u}) \hat{u}(\mathbf{k})^2 d^D \mathbf{k}. \quad (\text{A4})$$

Substituting this result in Eq. (38), we get the Landau equation (39).

## APPENDIX B: THE LENARD-BALESCU EQUATION AND THE TEMPORAL CORRELATION FUNCTION OF THE FORCE

In this Appendix, we give a short derivation of the Lenard-Balescu equation. Its derivation is classical but we shall need some intermediate steps in order to justify the expression (98) of the temporal correlation function. We follow the approach of Padmanabhan [3] but

we consider an arbitrary potential of interaction and we take into account collective effects. We start from the Klimontovich equation

$$\frac{\partial f_d}{\partial t} + \mathbf{v} \cdot \frac{\partial f_d}{\partial \mathbf{r}} - \nabla \Phi_d \cdot \frac{\partial f_d}{\partial \mathbf{v}} = 0, \quad (\text{B1})$$

where  $f_d(\mathbf{r}, \mathbf{v}, t) = \sum_i m \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{v} - \mathbf{v}_i)$  is the exact discrete distribution of particles and  $\Phi_d(\mathbf{r}, t)$  is the potential that they generate. Writing  $f_d = f + \delta f$  and  $\Phi_d = \Phi + \delta \Phi$  and taking the average of Eq. (B1), we get for a homogeneous distribution

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \langle \delta f \nabla \delta \Phi \rangle. \quad (\text{B2})$$

Subtracting Eqs. (B1) and (B2) and neglecting the quadratic terms (quasilinear approximation), we obtain

$$\frac{\partial \delta f}{\partial t} + \mathbf{v} \cdot \frac{\partial \delta f}{\partial \mathbf{r}} - \nabla \delta \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \quad (\text{B3})$$

It can be shown that the terms neglected are of order  $1/N^2$ . We can solve Eq. (B3) by Laplace-Fourier transform. This yields

$$(\omega - \mathbf{k} \cdot \mathbf{v}) \delta \hat{f} = -\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}} \delta \hat{\Phi}. \quad (\text{B4})$$

Therefore, the perturbed distribution function is given by

$$\delta \hat{f}(\mathbf{k}, \omega, \mathbf{v}) = -\frac{\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}}}{\omega - \mathbf{k} \cdot \mathbf{v}} \delta \hat{\Phi}(\mathbf{k}, \omega) + \hat{g}(\mathbf{k}, \omega, \mathbf{v}), \quad (\text{B5})$$

where  $\hat{g}$  is the general solution of  $(\omega - \mathbf{k} \cdot \mathbf{v}) \hat{g} = 0$ . For  $\hat{g} = 0$ , after integration over the velocity  $\mathbf{v}$ , we obtain the dispersion relation  $\epsilon(\mathbf{k}, \omega) = 0$  which arises in the linear stability analysis of the Vlasov equation (see Sec. II C). The condition of marginal stability corresponds to  $\epsilon(\mathbf{k}, 0) = 0$ . In the present context,  $\hat{g}$  is related to the discrete nature of the system, i.e. the fact that the exact distribution function  $f_d$  is a sum of  $\delta$ -functions. Therefore, its expression is given by

$$\hat{g}(\mathbf{k}, \omega, \mathbf{v}) = \frac{1}{(2\pi)^D} \sum_i m \delta(\mathbf{v} - \mathbf{v}_i) e^{i\mathbf{k} \cdot \mathbf{r}_i} \delta(\mathbf{k} \cdot \mathbf{v} - \omega). \quad (\text{B6})$$

On the other hand, using the fact that the relation between the potential  $\Phi$  and the distribution function  $f$  is a convolution, we have in Fourier space

$$\delta \hat{\Phi}(\mathbf{k}, \omega) = (2\pi)^D \hat{u}(\mathbf{k}) \int \delta \hat{f}(\mathbf{k}, \omega, \mathbf{v}') d^D \mathbf{v}'. \quad (\text{B7})$$

Combining Eqs. (B5) and (B7) we obtain

$$\delta \hat{\Phi}(\mathbf{k}, \omega) = (2\pi)^D \frac{\hat{u}(\mathbf{k})}{\epsilon(\mathbf{k}, \omega)} \int \hat{g}(\mathbf{k}, \omega, \mathbf{v}) d^D \mathbf{v}, \quad (\text{B8})$$

where we have introduced the dielectric function (10). It is now possible to compute the collision current

$$\langle \delta f \nabla \delta \Phi \rangle = \int \langle \delta \hat{f}(\mathbf{k}, \omega, \mathbf{v}) \delta \hat{\Phi}(\mathbf{k}', \omega') \rangle i \mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}') \cdot \mathbf{r}} e^{-i(\omega+\omega')t} d^D \mathbf{k} d^D \mathbf{k}' d\omega d\omega'. \quad (\text{B9})$$

Using Eqs. (B5) and (B8), we get

$$\begin{aligned} i \mathbf{k}' \langle \delta \hat{f}(\mathbf{k}, \omega, \mathbf{v}) \delta \hat{\Phi}(\mathbf{k}', \omega') \rangle &= -(2\pi)^{2D} \frac{\hat{u}(\mathbf{k}) \hat{u}(\mathbf{k}')}{\epsilon(\mathbf{k}, \omega) \epsilon(\mathbf{k}', \omega')} \frac{i}{\omega - \mathbf{k} \cdot \mathbf{v}} \mathbf{k}' \left( \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}} \right) \\ &\times \int \langle \hat{g}(\mathbf{k}, \omega, \mathbf{v}') \hat{g}(\mathbf{k}', \omega', \mathbf{v}'') \rangle d^D \mathbf{v}' d^D \mathbf{v}'' + (2\pi)^D \frac{i}{\epsilon(\mathbf{k}', \omega')} \hat{u}(\mathbf{k}') \mathbf{k}' \int \langle \hat{g}(\mathbf{k}, \omega, \mathbf{v}) \hat{g}(\mathbf{k}', \omega', \mathbf{v}'') \rangle d^D \mathbf{v}''. \end{aligned} \quad (\text{B10})$$

From Eq. (B6), the correlation function is given by

$$\langle \hat{g}(\mathbf{k}, \omega, \mathbf{v}') \hat{g}(\mathbf{k}', \omega', \mathbf{v}'') \rangle = \frac{m}{(2\pi)^D} f(\mathbf{v}') \delta(\mathbf{v}' - \mathbf{v}'') \delta(\mathbf{k} \cdot \mathbf{v}' - \omega) \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega'). \quad (\text{B11})$$

Substituting these expressions in Eq. (B9) and using

$$\epsilon(\mathbf{k}, \omega) = 1 + (2\pi)^D \hat{u}(\mathbf{k}) \int \left[ \frac{P}{\omega - \mathbf{k} \cdot \mathbf{v}} - i\pi \delta(\omega - \mathbf{k} \cdot \mathbf{v}) \right] \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}} d^D \mathbf{v}, \quad (\text{B12})$$

where  $P$  denotes the principal part, we finally obtain the Lenard-Balescu equation (49). Moreover, from the above expressions, it is easy to obtain the following expression for the correlations of the potential

$$\langle \delta \hat{\Phi}(\mathbf{k}, \omega) \delta \hat{\Phi}(\mathbf{k}', \omega') \rangle = m(2\pi)^D \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega') \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \omega)|^2} \int f(\mathbf{v}') \delta(\mathbf{k} \cdot \mathbf{v}' - \omega) d^D \mathbf{v}'. \quad (\text{B13})$$

Taking the inverse Fourier transform for the time variable, we obtain

$$\langle \delta \hat{\Phi}(\mathbf{k}, 0) \delta \hat{\Phi}(\mathbf{k}', t) \rangle = m(2\pi)^D \delta(\mathbf{k} + \mathbf{k}') \hat{u}(\mathbf{k})^2 \int \frac{1}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}')|^2} e^{-i\mathbf{k} \cdot \mathbf{v}' t} f(\mathbf{v}') d^D \mathbf{v}'. \quad (\text{B14})$$

Noting that the force acting on the particle at time  $t$  is

$$\mathbf{F}(t) = - \int i \mathbf{k} \delta \hat{\Phi}(\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{v} t} d^D \mathbf{k}, \quad (\text{B15})$$

we finally obtain the temporal correlation function of the force in the form

$$\langle F^\mu(0) F^\nu(t) \rangle = m(2\pi)^D \int k^\mu k^\nu \frac{\hat{u}(\mathbf{k})^2}{|\epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}')|^2} e^{-i\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}') t} f(\mathbf{v}') d^D \mathbf{v}' d^D \mathbf{k}. \quad (\text{B16})$$

This expression generalizes Eq. (91) in the case where collective effects are properly accounted for.

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- [50] This assumes that, at each time  $t$ , the distribution function  $f(\mathbf{v}, t)$  is a stable stationary solution of the Vlasov equation. Therefore, we study how the velocity distribution of a spatially homogeneous system changes due to “collisions” (finite  $N$  effects). This implicitly assumes that the energy is higher than the critical energy  $E_c$  at which the homogeneous phase becomes unstable at statistical equilibrium (see Paper I).